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Regularity for a Fractional Liquid Crystal Model with Anomalous Dissipation and Thermal Effects

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Abstract

In this paper, we present proofs of the regularity criterion for weak solutions to a generalized liquid crystal model with fractional diffusion and thermal effects. Specifically, we prove that if certain decay estimates hold, the solutions (u, d, θ), representing the velocity field, the orientation vector in the crystal and the temperature field respectively, are regular at the origin. We employ the extension technique for the fractional Laplacian and establish decay estimates to demonstrate the boundedness and regularity of the solutions. Our results aim to extend classical theories by introducing anomalous dissipation and thermal influences, leading to potentially increase the framework for the analysis of liquid crystal models under varying conditions.

Keywords Liquid crystal model · Fractional diffusion · Thermal effects · Anomalous dissipation · Regularity · Weak solutions · Extension technique · Decay estimates

1 Introduction and Problem Description

The study of liquid crystals has been a significant area of research due to their properties and numerous applications in displays, sensors, and other technologies. The classical models of liquid crystals, such as the Ericksen-Leslie theory [1, 2], provide a framework for understanding the behavior of these materials. However, recent advancements have highlighted the need to incorporate additional factors potentially leading to fractional diffusion and thermal effects to capture the complex dynamics more accurately and with terms that can support additional applications and discussions (for example, a fractional-based diffusion).

In this work, we introduce a variation of the liquid crystal model that introduced fractional diffusion and thermal effects. This model extends the classical Ericksen-Leslie theory and provides additional descriptions of the behavior of liquid crystals under various conditions. The equations governing the model are as follows:

$$
\begin{aligned}
&\partial_t u + u \cdot \nabla u + \mu(-\Delta)^{\alpha} u + \nabla P = -\lambda \operatorname{div}(\nabla d \otimes \nabla d) + \kappa \nabla\theta,\\
&\partial_t d + u \cdot \nabla d + \gamma(-\Delta)^{\beta} d = -f(d) + \sigma\Delta\theta,\\
&\operatorname{div} u = 0,\\
&\partial_t \theta + u \cdot \nabla\theta - \nu\Delta\theta = -\xi(\nabla \cdot u),\\
&u(x,0) = u_0(x),\ d(x,0) = d_0(x),\ \theta(x,0) = \theta_0(x),
\end{aligned}
\tag{1}
$$

where $(x, t) \in \mathbb{R}^3 \times (0, T)$. Even though a liquid crystal in practical scenarios exists within a finite domain, modeling such systems in the whole space $\mathbb{R}^3$ can be justified for several reasons. Firstly, solving the complex system of PDEs on $\mathbb{R}^3$ often simplifies the mathematical treatment, avoiding the additional complexities introduced by boundary conditions in finite domains. This simplification can make the analysis more tractable. Secondly, modeling in $\mathbb{R}^3$ can be seen as an idealization that approximates the behavior in large but finite domains, particularly when boundary effects are negligible or the domain is significantly larger than the characteristic length scales of interest. Lastly, such an approach is useful in theoretical studies to derive general properties, stability criteria, and asymptotic behaviors that can later be adapted to more realistic finite domains. It shall be noted that in the system (1), u represents the velocity field of the liquid crystal, d is the orientation vector field of the liquid crystal molecules, θ is the temperature field, and P is the pressure field. The

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