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Modeling Loss Index Triggers for Catastrophe (Cat) Bonds: An Alternative Continuous Approach

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Title

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Abstract

This paper proposes a method for continuous-time random modeling of loss index-triggered catastrophe bonds (cat bonds) that simplifies both rating and pricing throughout their maturity period. This index is based on the amount of declared losses calculated as the difference between the total amount of the catastrophe and that of incurred-but-not-yet-reported losses, which is modeled by means of a geometric Wiener process. The fundamental assumption of this model lies in considering that this amount decreases proportionally to a function, hereby called the mixed-rate of claims statement, which represents the pace of claim statements as growing linearly up to a certain moment, after which it becomes constant until the bond reaches maturity.

Key words

Catastrophe bonds (cat bonds); reported loss amount; incurred-but-not-yet-reported loss amount; mixed claim reporting rate; geometric Brownian motion; catastrophic loss ratio.

The incurred-but-not-yet-reported loss amount is the fundamental variable of the model

1. Introduction

Cat bonds are financial assets that condition their coverage on the occurrence of a certain trigger that is established at the time of issue. This trigger is selected based on the risks covered and the way in which the indemnity process is structured, in an attempt to, from the perspective of the investor, maximize its transparency and, from the perspective of the sponsor, minimize the basis risk or that of insufficient coverage. For this reason, over time, the triggers used in the processes of securitization of insured risk have varied from the initial indemnity and parametric indexes triggers to the current trend of sector loss indexes (or loss indexes from the insurance industry). This is basically due to the fact that compared to the accounting book-supported structure for the indemnity trigger, the loss index triggers are easier to understand for the investor and reduce the moral hazard. Furthermore, from the perspective of the insurer, they prevent a lot of the information subject to confidentiality from being made public. However, the main drawback to this system of structuring cat bonds is related to the use of poorly developed indexes that do not accurately represent the industry losses and generate basis risk.

A relevant aspect in both the theoretical and practical analysis of these financial-actuarial instruments with loss index triggers is their pricing over a set time horizon, based on the definition of a model that makes it possible to calculate the dynamics of the total amount of the losses, and therefore, the claims ratio underlying this type of contracts.

Different authors have considered this matter. Cummins and Geman (1995) and Geman and Yor (1997) have developed models for assessing options and futures for catastrophic risks based on two hypotheses: they use, on the one hand, Wiener's geometric processes to describe the instantaneous claim reporting and, on the other, Poisson processes, which incorporate the possibility of the occurrence of large catastrophes in the model. Aase (1999; 2001) models the dynamics of the loss index through a Poisson process consisting of random jumps to assess futures and future options for catastrophes (*cat futures* and cat *options*), as a specific case of the model developed by Embrechts and Meister (1995), which represents the behavior of the underlying value by means of a combination of compound Poisson processes and a random claim frequency. Loubergé, Kellezi and Gilli (1999) apply the option valuation model for catastrophes developed by Cummins and Geman (1995) to calculate the price of a catastrophe bond whose trigger is a loss index from the insurance industry. Lee and Yu (2002) incorporate the credit risk in the valuation of cat bonds through a Brownian geometric movement, as well as practical factors associated with moral hazard and base risk. Cox and Pedersen (2000) propose a method for calculating the price of a cat bond in incomplete markets based on the definition of a certain term structure for the interest rates and structure of probabilities of the catastrophic risk occurring. Muermann (2003) uses the loss index modeling developed by Aase (1999) to make a valuation, consisting in actuarial terms of derivative assets, options and futures negotiated on the Chicago Board of Trade (hereafter, CBOT). Nowak and Romaniuk (2013) apply term structure of interest rates models (risk-free spot interest rates) under the hypothesis that the occurrence of the catastrophe is independent of the behavior of the financial markets. Finally, Zong-Gang and Chao-Qun (2013) consider a stochastic interest rate environment to describe catastrophic losses through a compound non-homogeneous Poisson process.

This review of the financial-actuarial literature reveals the frequent use of the Brownian geometric movement to model the behavior of the loss index triggering the derivatives associated with insurance in general, and more specifically within this type of assets, cat bonds. Working under this hypothesis leads us to assume exponential growth, on average, of the instantaneous claim reporting, which contrary to what the empirical evidence reveals,

The incurred-but-not-yet-reported loss amount decreases according to a Brownian geometric movement

tends to be uniform over time. To resolve these inconsistencies, Alegre, Pérez-Fructuoso and Devolder (2003) developed a random model over discrete time that models the behavior of the loss index underlying the futures and options for catastrophic risks negotiated on the CBOT, based on the definition of the total amount of a catastrophe as the sum of two random variables: the reported loss amount and the incurred-but-not-yet-reported loss amount, limiting the possibility of the occurrence of catastrophes to one per period. Later, Pérez-Fructuoso (2008; 2009) expanded the previous discrete random model to the continuous field, assuming that the dynamics of the amount of incurred-but-not-yet-reported loss amount follows a Brownian geometric movement representative of a time decrease of this variable according to a real function of the real variable, referred to as the “rate of claim statements”.

For the case of a constant rate of claim statements (instantaneous rate of claim statements), the reported loss amount, the fundamental variable for the calculation of the loss index in the case of the occurrence of a catastrophe, is thus easily obtained as the difference between the total amount of the catastrophe and the incurred-but-not-yet-reported loss amount. However, disrupting the rate of claim statements with white noise amplified by volatility can result in negative values for said rate, which would cause an increase in the time of the incurred-but-not-yet-reported loss amount, due to the inverse variation defined for said variable. This can happen when, after the claim statements have been made, the appraisal by the insurance adjusters results in loss valuations that are less than those initially estimated. For this reason, the incorporation of randomness through a Wiener process would only be valid for volatility values that would give rise to a practically negligible probability that the incurred-but-not-yet-reported loss amount would increase.

In order to solve this problem, this article proposes a continuous model based on the same hypotheses as the original model by Pérez-Fructuoso (2008; 2009); i.e., the incurred-but-not-yet-reported loss amount decreases according to a function called the “rate of claim statements,” which is disrupted by means of a Wiener process that reflects the irregularity of the claim statements over time. The essential difference from the preceding model is based on the mixed definition of the rate of claim statements, i.e., as a defined function in two segments: the first of which grows linearly until a point in time when the pace of claim statements changes and becomes constant at a certain level.

The article is structured as follows: after an exhaustive review of the cat bond pricing models developed to date, section 2 presents the definition of the risk coverage instruments analyzed and includes an example of how they work. Section 3 presents the hypotheses on the occurrence of catastrophes and loss reporting upon which the loss index will be formulated subjacent to the bond and based on which their valuation will be carried out by means of traditional methods to assess the options used in the capital markets. Next, the process is carried out to calculate the variable corresponding to the incurred-but-not-yet-reported loss amount, first in a deterministic context and then randomly, assuming that it decreases according to a function called the “rate of claim statements,” which is disrupted using a Wiener process that reflects the irregularity of the claim statements over time. The reported loss amount, the value of which will be equal to the loss index in the event the catastrophe occurs, is obtained by the difference between the total amount of the catastrophe and the incurred-but-not-yet-reported loss amount. These magnitudes are calculated for any functional definition of the rate of claim statements and considering that the pace of claim statements, represented by the rate of claim statements, is a function defined in segments, which increases until a certain point and then becomes constant until the bond reaches maturity. Section 4 focuses on the calculation of the loss index. Finally, section 5 summarizes the most important conclusions reached in this work.

Harvard Deusto Business Research. Volume VI. Issue 2. Pages 84-101. ISSN: 2254-6235

2. Contextualization of cat bonds

2.1. Definition of a cat bond and current market situation

Cat bonds were created in the early 1990s to give the insurance industry access to a new source of risk coverage through the capital markets. Even though their structure is similar to that of traditional bonds, the results of catastrophe bonds are conditioned by the occurrence of a certain triggering event, the parameters of which are established at the time of issue (Pérez-Fructuoso, 2005).

For the most part, these instruments are sponsored by insurance companies, reinsurers and private companies that transfer to a special purpose vehicle (SPV) all or part of their catastrophic subscription risk. The SPV, in return, takes out a conventional re-insurance policy with the sponsor and seeks out financing (issuing bonds) in the capital market, which in turn acts as the counterparty in the established reinsurance agreement. The cash flows obtained with the bond issue and the premium paid by the cedent by way of the reinsurance price are invested by the SPV in short-term assets with high profitability that are deposited in a collateral account, insuring the transaction and generating sufficient resources to cover the risks undertaken in the reinsurance contract and the coupon payment promised to investors in the amount loaned through the purchase of the bonds. The real benefits generated in this account are exchanged at the LIBOR, with a swap counterparty that is highly rated by the rating agencies. Through this swap mechanism, the bonds become floating rate notes, so that the interest rate risk is eliminated for the most part. During the life of the bond, the periodic interests paid by the SPV to investors are obtained from the combination of two components: the premiums paid by the sponsor by way of reinsurance coverage and the LIBOR profitability generated by the principal of the bond, which is guaranteed by the counterparty of the swap. Then, at the end of the life of the bond, if the claim covered by the contract does not occur, the principal is returned to the investors, just like with any other fixed income investment. However, if the triggering claim event of the bond occurs, depending on its structure and the reinsurance contract, the investors will lose the interest and the principal of the investment or part of them.

Currently, the main investors in cat bonds are, in order of importance, large multinational companies (40%), life insurance companies (20%), hedge funds (15%), investment funds (10%), reinsurers (10%) and banks (5%). By region, the United States invests the most in this type of instruments (59%), followed by Europe (25%), Bermuda (11%), Japan (3%), Canada (1%) and Australia (1%).

In 2015, $5.917 billion of cat bonds were issued in a total of 25 transactions, a figure that is lower than the record $7.9 billion reached in 2014. During the first nine months of 2016, activity in the cat bond market reached $3.7 billion.

Table 1
Capital risk issued in cat bonds between 1998 and 2015 (in millions of dollars)

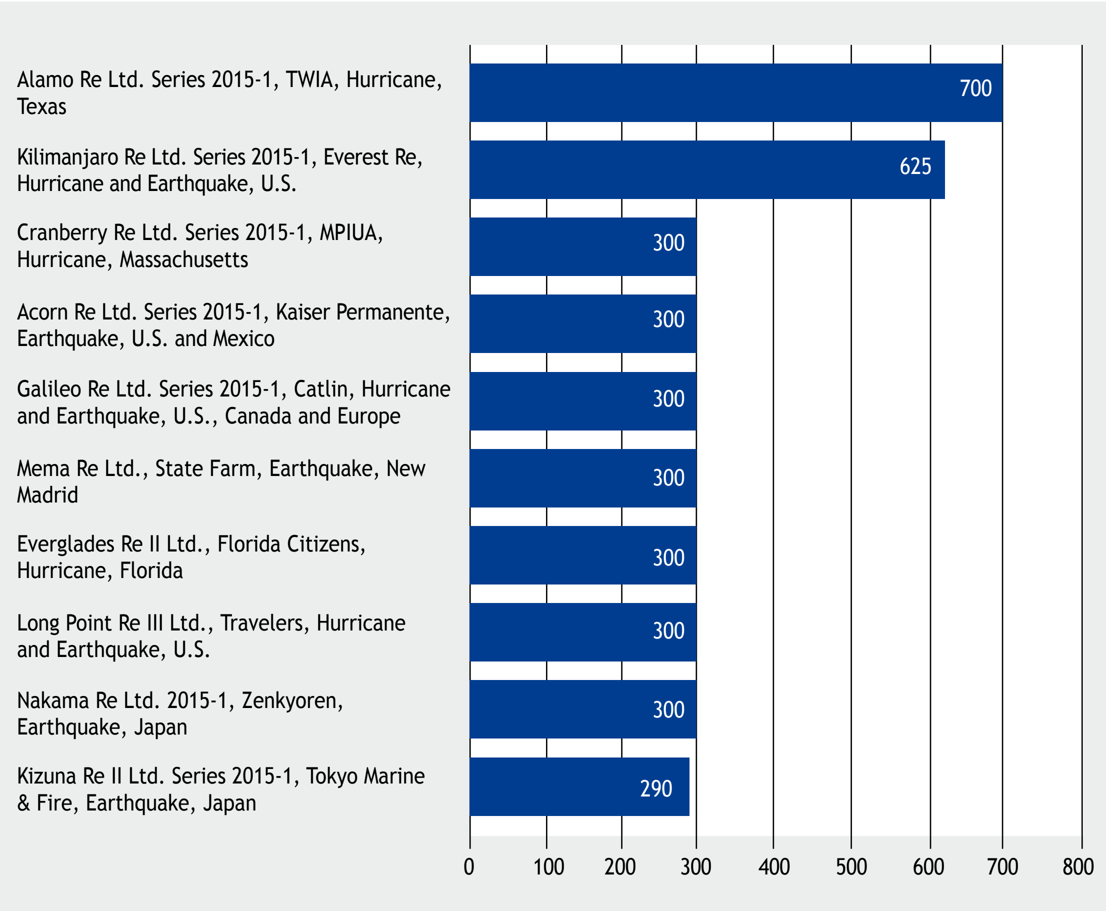
1998	1999	2000	2001	2002	2003	2004	2005	2006
874.2	1,052.5	1,142	966.9	989.5	1,988.2	1,142.8	1,499	4,614.7
2007	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**
7,187	3,009.9	3,396	4,599.9	4,107.1	5,855.3	7,083	7,926.7	5,917.2

Source: author's own work, based on Artemis (2016).

As can be seen in Table 1, since 1998, when the cat bond market began to be more active, the issued capital risk figure has shown an increasing trend, with a drop in 2008 coinciding with the start of the worst financial crisis in history. Since 2009, however, the issue of cat bonds has quickly increased, reaching $7.083 billion in 2013, with 31 transactions, and the record figure of $7.9 billion in 2014, above the figure of $7.2 billion issued in 2007. The most important transactions made with cat bonds in 2015 are shown in Table 2.

Table 2

Main transactions with cat bonds in 2015 (in millions of dollars)



Source: Insurance Information Institute.

In terms of the trigger type of cat bonds, until 2007, the initial issues based on indemnity triggers gave way to a growing preference for contracts with insurance industry loss index triggers that included the occurrence of catastrophic damage associated with a certain catastrophe. In 2007, however, the use of indemnity triggers resurfaced, evidencing the increasing sophistication of investors and the growing leverage of the sponsors. This trend has been maintained until present. Table 3 shows the evolution of cat bond issues since 2007 according to the type of trigger.

The rate of claim statements is defined as a mixed function

Table 3
Cat bonds issued by type of trigger (percentage)

	Indemnity	Loss index	Parametric	Modeled losses	Multiple trigger
2007	35.1	27.3	20.2	6	5
2008	51.7	20.4	24.4	0	0
2009	25.7	38.9	19.9	8.7	6.8
2010	34.7	37.3	4.4	2.7	12.6
2011	30.1	48.8	4.1	4.8	0
2012	49	23.4	9.8	5.9	3.2
2013	58	21.9	7.8	0	5.5
2014	67.7	22.1	0.7	2.8	1.1
2015	57.8	16.3	12.6	0	2.6
2016	68.7	26.5	0.6	0	0

Source: author's own work, based on Artemis (2016).

2.2. Example of how a cat bond works: cat bond issued by the United Services Automobile Association (USAA)

The United Services Automobile Association (USAA) is a non-life insurance company with headquarters in San Antonio that offers personal financial management products to military service members (on active duty or retired) belonging to the U.S. Army and their dependents (Cox, Fairchild & Pedersen, 2000; Pérez-Fructuoso, 2005).

To cover the overexposure to the risk of hurricanes in its automobile and real estate portfolios, for the first time in 1997 it issued annual term bonds through a reinsurance captive from the Cayman Islands, Residential Re, with a face value of $477 million and variable coupon payment and principal return, depending on the losses recorded by the real estate properties belonging to their insured parties, caused by hurricanes in the Gulf of Mexico and on the East Coast of the United States. The risk covered in the issue is defined precisely as "damage to USAA clients on the East Coast and the Gulf of Mexico of the United States as the result of the occurrence of a hurricane during the period between June 1997 and June 1998, classified as a category 3, 4 or 5 storm on the Saffir-Simpson scale, in the states covered by the contract and the losses of which exceed $1 billion." The bonds were issued in two series. In the first series, only the coupons had any risk associated with them; the principal was guaranteed. For the second series, both the coupons and the principal were conditioned by the risk of hurricanes. In this manner, in the second series, the coupons and the principal of the bond are not paid to investors if the losses derived from the risk defined in the contract exceed $1 billion. For the first series, on the other hand, after losses above $1 billion, the coupon begins to decrease and after $1.5 billion, the coupon disappears entirely.

For the series with only the risk coupon, the profitability offered by the bond is the LIBOR plus a differential of 2.73% (273 basis points). The series with the risk coupon and principal offers a yield of the LIBOR plus 5.76% (576 basis points).

How was the transaction carried out? Residential Re reached a reinsurance agreement with USAA to cover 80% of the level of $500 million in excess of the first $1 billion of losses by USAA. USAA retained the remaining 20% of the $500 million, i.e., $100 million. Residential

Harvard Deusto Business Research. Volume VI. Issue 2. Pages 84-101. ISSN: 2254-6235

Table 4
Specifications of the cat bond issued by the USAA

Issuer	Residential Re Ltd. (Cayman Islands): SPV company from the Cayman Islands that provides reinsurance to the USAA.
Reinsured	USAA
Investors	Investment funds, life insurance companies, reinsurers, etc.
Issued bond coupon	• Class A-I bonds: LIBOR + 273 points. • Class A-II bonds: LIBOR + 576 points.
Reinsurance agreement	Residential agrees with USAA to cover 80% of $500 million of risk in excess of the first $1 billion of losses. USAA retains the remaining 20% of the $500 million tier ($100 million).
Covered risk/Triggering event of the contract	Hurricane occurring between June 1997 and June 1998, classified as category 3, 4 or 5 on the Saffir-Simpson scale in the states covered by the contract and in which losses exceed $1 billion.
Type of coverage	Single occurrence: the contract offered by Residential is limited to a single hurricane that causes losses in an amount greater than $1 billion. If another hurricane occurs that produces similar losses, USAA is not covered.
Covered states	Alabama, North Carolina, South Carolina, Connecticut, Delaware, District of Columbia, Florida, Georgia, Louisiana, Maine, Maryland, Massachusetts, Mississippi, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, Vermont and Virginia.
Period of coverage	June 16, 1997 to June 14, 1998.
Bonds	Two types of bonds are issued: • Class A-I: $164 million ($77 million of principal protected; the rest variable). • Class A-II: $313 million (principal is 100% variable).
Credit rating	• Class A-I: AAAr/Aaa/AAA/AAA by S&P, Moody's, Fitch and D&P, respectively. • Class A-II: BB/BA/BB/BB by S&P, Moody's, Fitch and D&P, respectively.
Risk assessment and modeling	Applied Insurance Research, Inc. (AIR): company that has applied its hurricane simulation model to assess the bond risk.

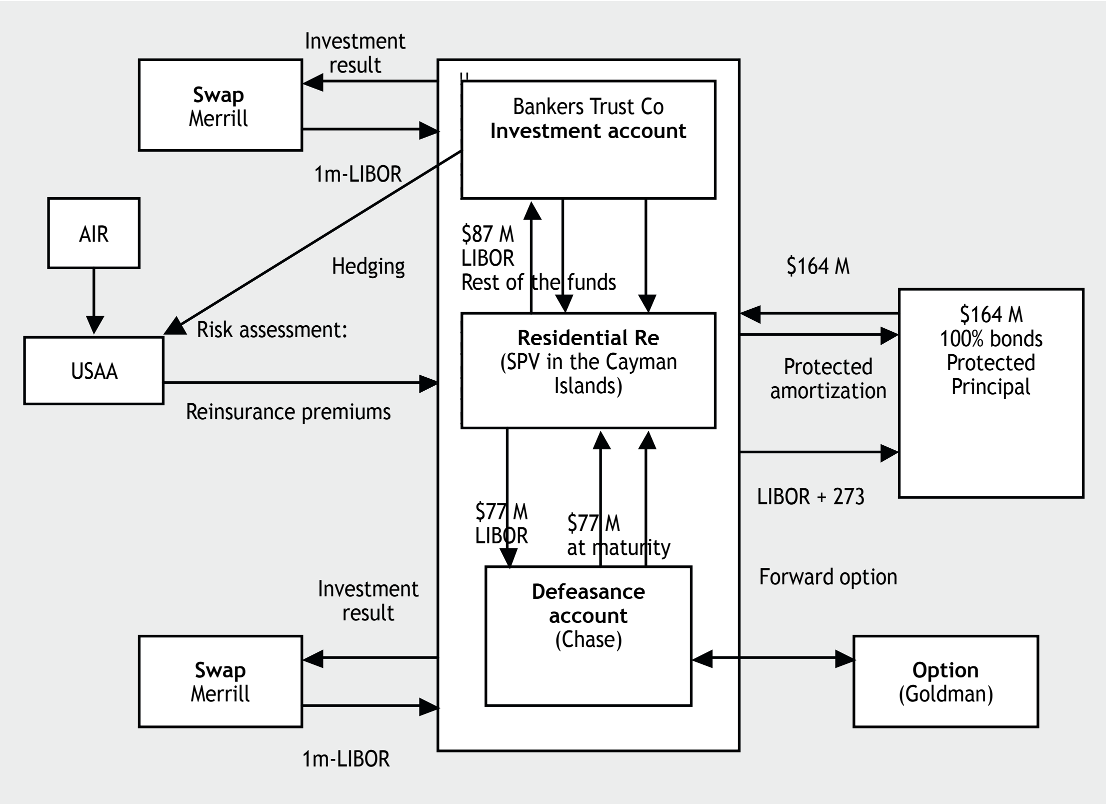
Re issued two types of bonds, Class A-I and Class A-II, with variable interest coupons, for a total of $476.98 million. Approximately $400 million (80% of the coverage of $500 million) were made available to USAA in the case that the covered claim were to occur. The remaining 77 million were placed in a type of investment account called a defeasance account. The functioning of the Class A-I series is shown in Table 5.

Class A-I bonds were rated higher by the rating agencies because the principal was protected and therefore its return was assured. 53% of the capital obtained with the issue of this series, $87 million, was invested by Bankers Trust Co as a secured debt. The rest, approximately $77 million, was deposited in an account that was also invested in secured debt, but in this case, it was administered by Chase Manhattan. To guarantee the profitability obtained in these two accounts, a swap was conducted with Merrill Lynch Capital Service, as it was in the case of the Class A-II bond, thus eliminating the interest rate risk.

Since the reinsured event did not occur, at bond maturity, the principal was returned to the investors. If the claim trigger would have occurred that is described in the contract, the collateral account funds would have been used to purchase U.S. Treasury bonds with a ten-year maturity in the amount of $163.8 million (the principal of the Class A-I bonds). This purchase was guaranteed by the irrevocable agreement with Goldman Sachs Mitsui Marine Derivate Products, L.P. in the form of a forward option agreement that insured the purchase

The rate of claim statements increases linearly until s_m^i and then remains constant

Table 5
Structure of the series A-I cat bond issue by USAA



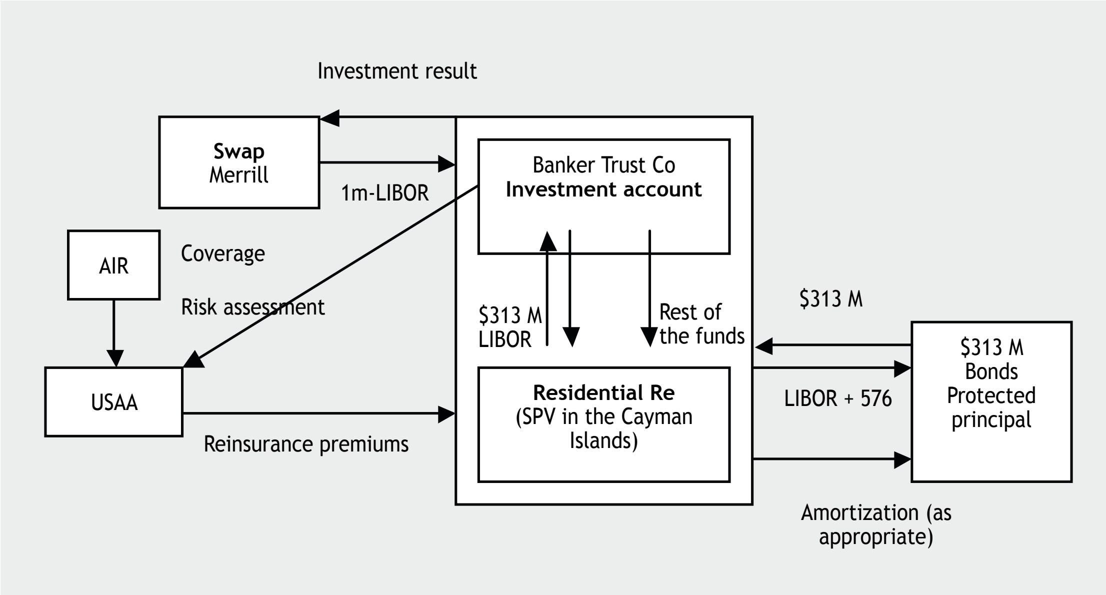
Source: Pérez-Fructuoso (2005).

of the bonds that permitted the return of the principal at a specified price (a contingent defeasance securities agreement, or an operation to establish a risk-free bond portfolio, normally consisting of state debt, that will be used for the payment of the *cash-flow* promised to a group of creditors). In this case, the maturity period would have changed to December 15, 2008 instead of June 15, 1998.

For series Class A-II, the functioning scheme is shown in Table 6.

The $313 million obtained with the issue of Class A-II bonds was invested by Bankers Trust Co in secured debt. The interest (or coupons) for the bond purchasers was obtained from the monthly reinsurance premiums paid by USAA, from the earnings from the investment account and a swap agreement signed with Merrill Lynch Capital Service. This agreement was established as the result of the current legal restrictions, and as a result of it, the earnings from the secured debt from Bankers Trust Co were exchanged with monthly LIBOR interest to insure the payment of the variable interest promised to the bond investors (LIBOR + 576 points). Since the reinsured event did not occur, at bond maturity on June 15, 1998, the principal was returned. If the trigger event would have occurred, the interest and principal would have been canceled out proportionally to the claim paid.

Table 6
Structure of the series A-II cat bond issue by USAA



Source: Pérez-Fructuoso (2005).

Taking this operation into account, USAA, using the capital market, has had access to more than $6 billion since 1997, through 25 different catastrophe bond offers. The last issue sponsored by USAA occurred in December 2015 in the amount of $125 million to cover multiple risks through a cat bond designed with an indemnity trigger.

3. Model for calculating the loss index

3.1. Catastrophe occurrence hypothesis

According to Pérez-Fructuoso (2008), we establish $[0, T] \subset [0, T']$ as the risk period of the catastrophe bond, in such a way that $T' \geq T$ is the maturity date or date of contract amortization and $\tau \in [0, T]$ is the moment when the disaster occurs that is covered by the bond issue. We define K_τ^i as the random variable that represents the total amount of the disaster with intensity i occurring at moment τ with $i = 1, 2, 3$, in such a way that $i = 1$ if the disaster that occurs is small in amount, $i = 2$ if it is of an average amount and $i = 3$ if it is large in amount (Alegre, Pérez-Fructuoso & Devolder, 2003).

Finally, we consider a variable indicator, δ_τ^i, which has a value of 0 if the catastrophe does not occur that is covered by bond at time $\tau \in [0, T]$; otherwise it is 1.

3.2. Claim reporting hypothesis

We assume that the claim reporting process associated with the occurrence of a catastrophe begins the very moment that it occurs and lasts until the moment of bond maturity, T'. Therefore, for a valuation point $t \in (\tau, T'] \subset [0, T']$, we define the total amount of the catastrophe occurred at time point τ, K_τ^i as the sum of two random variables —see formula 1, in which $S_\tau^i(t)$ is the reported loss amount and $R_\tau^i(t)$ is the incurred-but-not-yet-reported loss amount, both referring to valuation point t–.

The loss index is the reported loss amount, multiplied by a variable indicator

$$K_\tau^i = S_\tau^i(t) + R_\tau^i(t) \quad (1)$$

The variables $R_\tau^i(t)$ and $S_\tau^i(t)$ are subject to the following boundary conditions:

a) Initial boundary condition, $t = \tau$: if the moment of bond valuation coincides with the moment the disaster occurs, $R_\tau^i(t) = K_\tau^i$ and $S_\tau^i = 0$

the incurred-but-not-yet-reported loss amount matches the total volume of the catastrophe and, consequently, the reported loss amount is zero.

b) Final boundary condition, $t \to \infty$: if the valuation of the bond occurs at a time distant enough from the occurrence of the catastrophe (tends to infinity), $\lim_{t \to \infty} R_\tau^i(t) = 0$ and $\lim_{t \to \infty} S_\tau^i(t) = K_\tau^i$

the losses associated with the catastrophe have already been fully reported and therefore there are no incurred-but-not-yet-reported losses.

3.3. General calculation of the variables $R_\tau^i(t)$ and $S_\tau^i(t)$ in the true model

Based on the analysis of the empirical evidence, a fundamental hypothesis of the model is considered to be that the intensity in the claim statements is very high immediately after the occurrence of the disaster and diminishes over time until disappearing when there are no longer any claims to declare. As a result of this, during an early construction phase of the model, the instantaneous claim rate is represented by a true differential equation (see equation 2) that describes an increase in the amount of claim statements proportional to the incurred-but-not-yet-reported loss amount and in which $\alpha_\tau^i(t-\tau)$ is a real function of the real variable referred to as the rate of claim statements, the explicit form of which is obtained through an analysis of the empirical data on catastrophe claim statements and according to the hypothesis that the claims associated with the average amount disasters will be reported sooner than the claims resulting from large disasters, in other words $\alpha_\tau^2(t-\tau) > \alpha_\tau^3(t-\tau)$. In terms of small catastrophes, $i = 1$, it is considered that they are reported instantly at the moment when they occur, and form part of the loss index directly. Therefore $\alpha_\tau^1(t-\tau) \to \infty$, $S_\tau^1(t) = K_\tau^1$ and $R_\tau^1(t) = 0$.

$$dS_\tau^i(t) = \alpha_\tau^i(t-\tau) R_\tau^i(t) dt \quad (2)$$

Differentiating equation 2, we get expression 3.

$$dS_\tau^i(t) = -dR_\tau^i(t) dt \quad (3)$$

In equation 2, by replacing $dS_\tau^i(t)$ with the result obtained in expression 3, we obtain the true differential equation that describes the evolution of the incurred-but-not-yet-reported loss amount, $R_\tau^i(t)$, a fundamental variable in our modeling, as is shown in equation 4.

Harvard Deusto Business Research. Volume VI. Issue 2. Pages 84-101. ISSN: 2254-6235

$$dR_{\tau}^{i}(t) = -\alpha_{\tau}^{i}(t-\tau)\, R_{\tau}^{i}(t)\, dt \qquad (4)$$

Solving ordinary differential equation 4 with the boundary conditions a) and b) that were previously defined, we obtain equation 5.

$$R_{\tau}^{i}(t) = K_{\tau}^{i} e^{-\int_{0}^{t-\tau} \alpha_{\tau}^{i}(s)\, ds} \qquad (5)$$

By replacing 5 in equation 1, which establishes the relationship between the variables $R_{\tau}^{i}(t)$ and $S_{\tau}^{i}(t)$, we easily obtain the reported loss amount up to t, $S_{\tau}^{i}(t)$ as the difference between the total amount of the catastrophe and the incurred-but-not-yet-reported loss amount, as shown in equation 6.

$$S_{\tau}^{i}(t) = K_{\tau}^{i} - R_{\tau}^{i}(t) = K_{\tau}^{i}\left[1 - e^{-\int_{0}^{t-\tau} \alpha_{\tau}^{i}(s)\, ds}\right] \qquad (6)$$

3.4. Calculation of the variables $R_{\tau}^{i}(t)$ and $S_{\tau}^{i}(t)$ in the true model for a mixed rate of claim statements

The model presented in the previous section is developed in the work by Pérez-Fructuoso (2008), following the work of other authors, such as Cummins and Geman (1995), for the random case and assuming a constant value for the rate of claim statements function. This hypothesis implies the assumption that the pace of claim statements is the same for the entire period analyzed. However, experience shows that the pace of claim statements is faster in the first few days following a disaster, which causes a greater reduction in the incurred-but-not-yet-reported loss amount at that time, and thus a greater growth in the amount of claim statements.

This article presents an alternative definition for this rate, which we call the “mixed rate of claim statements,” the expression of which is shown in equation 7 and represents a linear increase in the claim statements up to point s_{m}^{i} that then remains constant from this point until at level α_{τ}^{i}.

$$\alpha_{\tau}^{i}(s) = \begin{cases} \left(\dfrac{\alpha_{\tau}^{i}}{s_{m}^{i}}\right) s & 0 \le s \le s_{m}^{i} \\ \alpha_{\tau}^{i} & s > s_{m}^{i} \end{cases} \qquad (7)$$

Harvard Deusto Business Research. Volume VI. Issue 2. Pages 84-101. ISSN: 2254-6235

When the rate of claim statements is defined in this manner, it is possible to distinguish between two cases, depending on where the valuation point is:

1. If the valuation point, t, is before the point when the pace of claim statements changes, $\tau + s^i_m$, i.e., $\tau \leqslant t \leqslant \tau + s^i_m$, the solution for the integral in equation 5 is that which appears in equation 8.

$$\int_0^{t-\tau} \alpha^i_\tau(s)\,ds = \int_0^{t-\tau} \frac{\alpha^i_\tau}{s^i_m} s\,ds = \frac{\alpha^i_\tau}{s^i_m}\frac{s^2}{2}\Bigg]_0^{t-\tau} = \frac{\alpha^i_\tau (t-\tau)^2}{2s^i_m} \qquad (8)$$

In this case, the incurred-but-not-yet-reported loss amount in t and the reported loss amount up to moment t are those appearing in equations 9 and 10.

$$R^i_\tau(t) = K^i_\tau e^{-\alpha^i_\tau (t-\tau)\frac{(t-\tau)}{2s^i_m}} \quad \text{con } i = 2,3 \qquad (9)$$

$$S^i_\tau(t) = K^i_\tau \left[1 - e^{-\alpha^i_\tau (t-\tau)\frac{(t-\tau)}{2s^i_m}}\right] \quad \text{con } i = 2,3 \qquad (10)$$

2. If the valuation point, t, is after the point when the pace of claim statements changes, $\tau + s^i_m$, i.e., $t > \tau + s^i_m$, the solution for the integral in equation 5 is that which appears in equation 11.

$$\int_0^{t-\tau} \alpha^i_\tau(s)\,ds = \int_0^{s^i_m} \frac{\alpha^i_\tau}{s^i_m} s\,ds + \int_{s^i_m}^{t-\tau} \alpha^i_\tau ds = \frac{\alpha^i_\tau}{s^i_m}\frac{s^2}{2}\Bigg]_0^{s^i_m} + \alpha^i_\tau s\Big]_{s^i_m}^{t-\tau} = \alpha^i_\tau (t-\tau) - \frac{\alpha^i_\tau s^i_m}{2} \qquad (11)$$

In this case, the incurred-but-not-yet-reported loss amount in t and the reported loss amount up to moment t are those appearing in equations 12 and 13.

$$R^i_\tau(t) = K^i_\tau e^{-\alpha^i_\tau (t-\tau)} e^{\frac{\alpha^i_\tau s^i_m}{2}} \quad \text{con } i = 2,3 \qquad (12)$$

$$S^i_\tau(t) = K^i_\tau \left[1 - e^{-\alpha^i_\tau (t-\tau)} e^{\frac{\alpha^i_\tau s^i_m}{2}}\right] \quad \text{con } i = 2,3 \qquad (13)$$

Harvard Deusto Business Research. Volume VI. Issue 2. Pages 84-101. ISSN: 2254-6235

The claim statements are the total amount, minus the incurred-but-not-yet-reported loss amount

3.5. General calculation of the variables $R_\tau^i(t)$ and $S_\tau^i(t)$ in the random model

To capture the irregular behavior of the catastrophe claim statements over time, we introduced a Wiener process in equation 4, which led to equation 14, a stochastic differential equation (Pérez-Fructuoso, 2008; 2009) in which $\alpha_\tau^i(t-\tau)$ represents the trend of the process, α_τ^i is a constant indicating the volatility of the process and $w_\tau^i(t-\tau)$ is a standard Wiener process associated with the catastrophe of type i occurring at moment τ.

$$dR_\tau^i(t) = -\alpha_\tau^i(t-\tau)\,R_\tau^i(t)\,dt + \sigma_\tau^i R_\tau^i(t)\,dw_\tau^i(t-\tau) \quad \forall t \in [\tau, T] \qquad (14)$$

The Wiener process represents the differences in the claim reporting intensity, as it is considered that each disaster has its own characteristics that are not expressed in the model. This is reflected in the model by introducing different disturbances through independent Wiener processes.

Differential equation 14 is solved by applying Itô's lemma to the transformation $y = \ln R_\tau^i(t)$ (Arnold, 1974); from this, considering boundary conditions a) and b), we obtain equation 15:

$$R_\tau^i(t) = K_\tau^i e^{-\int_0^{t-\tau} \alpha_\tau^i(s)\,ds - \frac{(\sigma_\tau^i)^2}{2}(t-\tau) + \sigma_\tau^i w_\tau^i(t-\tau)} \qquad (15)$$

By replacing 15 in equation 1, which establishes the relationship between the variables $R_\tau^i(t)$ and $S_\tau^i(t)$, we easily obtain the reported loss amount until t, $S_\tau^i(t)$, as the difference between the total amount of the catastrophe and the incurred-but-not-yet-reported loss amount (see equation 16), without the need to define a stochastic differential equation to describe its dynamics.

$$S_\tau^i(t) = K_\tau^i - R_\tau^i(t) = K_\tau^i\left[1 - e^{-\int_0^{t-\tau} \alpha_\tau^i(s)\,ds - \frac{(\sigma_\tau^i)^2}{2}(t-\tau) + \sigma_\tau^i w_\tau^i(t-\tau)}\right] \qquad (16)$$

3.6. Calculation of the variables $R_\tau^i(t)$ and $S_\tau^i(t)$ in the random model for a mixed rate of claim statements

In this case, the incurred-but-not-yet-reported loss amount, $R_\tau^i(t)$, and the reported loss amount, $S_\tau^i(t)$, are obtained by replacing in equations 15 and 16 the results obtained in 8 and 11, and performing the operations, so that:

- If the valuation point is before the point when the pace of claim statements changes, $\tau + s_m^i$, i.e., $\tau \leq t \leq \tau + s_m^i$, equations 17 and 18 apply.

$$R_\tau^i(t) = K_\tau^i e^{-\left(\frac{\alpha_\tau^i}{2s_m^i}(t-\tau)+\frac{(\sigma_\tau^i)^2}{2}\right)(t-\tau)+\sigma_\tau^i w_\tau^i(t-\tau)} \quad \text{con } i = 2,3 \tag{17}$$

$$S_\tau^i(t) = K_\tau^i \left[1 - e^{-\left(\frac{\alpha_\tau^i}{2s_m^i}(t-\tau)+\frac{(\sigma_\tau^i)^2}{2}\right)(t-\tau)+\sigma_\tau^i w_\tau^i(t-\tau)}\right] \quad \text{con } i = 2,3 \tag{18}$$

- If the valuation point is after the point when the pace of claim statements changes, $\tau + s^i_m$, i.e., $\tau > \tau + s^i_m$, equations 19 and 20 apply.

$$R_\tau^i(t) = K_\tau^i e^{-\left(\alpha_\tau^i+\frac{(\sigma_\tau^i)^2}{2}\right)(t-\tau)+\sigma_\tau^i w_\tau^i(t-\tau)} e^{\frac{\alpha_\tau^i s_m^i}{2}} \quad \text{con } i = 2,3 \tag{19}$$

$$S_\tau^i(t) = K_\tau^i \left[1 - e^{-\left(\alpha_\tau^i+\frac{(\sigma_\tau^i)^2}{2}\right)(t-\tau)+\sigma_\tau^i w_\tau^i(t-\tau)} e^{\frac{\alpha_\tau^i s_m^i}{2}}\right] \quad \text{con } i = 2,3 \tag{20}$$

4. Determination of the catastrophe loss index

A catastrophe loss index can be defined as the ratio between the total amount of losses associated with one or more catastrophes occurring throughout a specified period of time and a constant value whose definition depends on the type of index used (for example, this might be the volume of premiums accrued during the risk period to cover the related catastrophic losses or a constant value to refer to the losses recorded at market trading points).

Cat bonds that use loss indexes as triggers for indemnities only consider for the development of said indexes the occurrence of a catastrophe, and when making payments, they are based on the value that the index used reaches at the time of contract termination, *T'*. We thus have equation 21, in which $LI(T')$ is the value of the loss index at maturity.

$$LI(T') = \delta_\tau^i \times \frac{S_\tau^i(T')}{cte} = \begin{cases} 0 & \text{si } \delta_\tau^i = 0 \\ \dfrac{S_\tau^i(T')}{cte} & \text{si } \delta_\tau^i = 1 \end{cases} \tag{21}$$

In equation 21, by replacing $S_\tau^i(t)$ for $t = T'$ with its generic expression given in equations 18 and 20, the value of said index at maturity, depending on the time when the change occurs in the pace of claim statements, is calculated as follows:

$$LI(T') = \delta_\tau^i \times \frac{S_\tau^i(T')}{cte} = \delta_\tau^i \times \frac{1}{cte} \times K_\tau^i \times \left[1 - e^{-\left(\frac{\alpha_\tau^i}{2s_m^i}(T'-\tau) + \frac{\left(\sigma_\tau^i\right)^2}{2} \right)(T'-\tau) + \sigma_\tau^i w_\tau^i (T'-\tau)} \right] \tag{22}$$

- If $T' \leq \tau + s_m^i$, we have equation 23.

$$LI(T') = \delta_\tau^i \times \frac{S_\tau^i(T')}{cte} = \delta_\tau^i \times \frac{1}{cte} \times K_\tau^i \times \left[1 - e^{-\left(\alpha_\tau^i + \frac{\left(\sigma_\tau^i\right)^2}{2} \right)(T'-\tau) + \sigma_\tau^i w_\tau^i (T'-\tau)} e^{\frac{\alpha_\tau^i s_m^i}{2}} \right] \tag{23}$$

$LI(T')$ Is random because $S_\tau^i(T')$ is a random variable: *a priori,* when the bond is issued, it is not known whether the catastrophe covered by it will occur or not, and therefore its amount and the time of its occurrence are also unknown.

The value of this loss index at maturity has been determined at the time of issuance of the coverage contract. Next, we will analyze how its probability distribution changes when, over time, instant $t \in [\tau, T']$ is reached and the available information on the claim statements reported so far is incorporated. To this end, $LI^*(T') = LI^*(T')/F_t$ is defined as a random conditioned variable that represents the total amount of losses reported as of T', with F_t being a filter that represents the possible history of interval $[\tau, t]$.

$LI^*(T') = LI^*(T')/F_t$ is obtained by calculating, first of all, the total amount of the losses reported at any time $t \in [\tau, T']$, $LI(t)$, which conditions $LI(T')$ (i.e., $LI(t) \cong F_t$).

It is not necessary to define a differential equation to obtain the reported loss amount

Therefore, taking into account whether the valuation period t is before or after the point when the pace of claim statements changes, we obtain the following:

- If $t \leqslant \tau + s^i_m$, we have equation 24.

$$LI(t) = \delta^i_\tau \times \frac{S^i_\tau(t)}{cte} = \delta^i_\tau \times \frac{1}{cte} \times K^i_\tau \times \left[1 - e^{-\left(\frac{\alpha^i_\tau}{2s^i_m}(t-\tau) + \frac{\left(\sigma^i_\tau\right)^2}{2} \right)(t-\tau) + \sigma^i_\tau w^i_\tau(t-\tau)} \right] \quad (24)$$

- If $t > \tau + s^i_m$, we have equation 25.

$$LI(t) = \delta^i_\tau \times \frac{S^i_\tau(t)}{cte} = \delta^i_\tau \times \frac{1}{cte} \times K^i_\tau \times \left[1 - e^{-\left(\alpha^i_\tau + \frac{\left(\sigma^i_\tau\right)^2}{2} \right)(t-\tau) + \sigma^i_\tau w^i_\tau(t-\tau)} e^{\frac{\alpha^i_\tau s^i_m}{2}} \right] \quad (25)$$

Once $LI(t)$ has been determined, it is incorporated in the variable $LI(T')$, which produces the following value for the conditioned loss index:

- If $t \leqslant \tau + s^i_m$, we have equation 26.

$$LI^*(T') = (\delta^i_\tau / F_t) \times \frac{1}{cte} \times \left[\begin{array}{l} L(t) + \\ + (K^i_\tau / F_t) \times \left[1 - e^{-\alpha^i_\tau(T'-\tau) + \frac{\left(\sigma^i_\tau\right)^2}{2}(T'-t) + \sigma^i_\tau \times w^i_\tau(T'-t) + \frac{\alpha^i_\tau s^i_m}{2} + \frac{\alpha^i_\tau(t-\tau)^2}{2s^i_m}} \right] \times \\ \times e^{-\left(\frac{\alpha^i_\tau(t-\tau)}{2s^i_m} + \frac{\left(\sigma^i_\tau\right)^2}{2} \right)(t-\tau) + \sigma^i_\tau \times w^i_\tau(t-\tau)} \end{array} \right] \quad (26)$$

- If $t > \tau + s^i_m$, we have equation 27.

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$$LI^{*}(T') = (\delta_{\tau}^{i}/F_{t}) \times \frac{1}{cte} \times \left[\begin{array}{l} L(t) + \\ + (K_{\tau}^{i}/F_{t}) \times \left[1 - e^{-\left(\alpha_{\tau}^{i} + \frac{(\sigma_{\tau}^{i})^{2}}{2}\right)(t-\tau) + \sigma_{\tau}^{i} \times w_{\tau}^{i}(t-\tau) + \frac{\alpha_{\tau}^{i} s_{m}^{i}}{2}} \right] \times \\ \times e^{-\left(\alpha_{\tau}^{i} + \frac{(\sigma_{\tau}^{i})^{2}}{2}\right)(T'-t) + \sigma_{\tau}^{i} \times w_{\tau}^{i}(T'-t)} \end{array} \right] \qquad (27)$$

Calculated in this manner, the loss index makes it possible to easily calculate the catastrophe bond price at a time *t* of its negotiation period, applying the general theory of option valuation (see for example Loubergé, Kellezi & Gilli, 1999, or Pérez-Fructuoso, 2008).

5. Conclusions

The continuous model proposed in this work makes it possible to simply calculate the loss index trigger of the cat bond, providing its valuation at a particular point in time. Unlike many of the preceding models (see, for example, Cummins & Geman, 1995 or Geman & Yor, 1997) that assume growth over time of the reported loss amount and represent said evolution through a Brownian geometric movement, the main hypothesis of the model presented here is the definition of the dynamics of the claim statements based on a growth that is proportional to the incurred-but-not-yet-reported loss amount. This amount is the fundamental variable in the process of formalizing the model, the decreasing time evolution of which we modeled through a Wiener geometric process. Once this variable has been determined, the total of the claim statements is obtained as the difference between the total amount of the catastrophe and the incurred-but-not-yet-reported loss amount, thus eliminating the need to define a stochastic differential equation to describe its dynamics. The catastrophic loss index is the result of multiplying the reported loss amount by a random dichotomous variable that indicates whether or not the catastrophe has occurred.

These variables have been obtained in a generic manner, for any functional definition of the rate of claim statements, and then a mixed form has been attributed to said rate, which increases linearly until a certain point and then remains constant until the bond reaches maturity. The aim of considering this definition for the claim reporting rate as an alternative to the constant definition proposed in the preceding models is to attempt to better represent the real evolution of the claim statements over time and to eliminate the incongruence that emerged in the initial model when said rate was disrupted by the incorporation of the Wiener process.

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