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Analysis of travelling waves and propagating supports for a nonlinear model of flame propagation with a p-Laplacian operator and advection

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Abstract

In this paper, we propose a new model to characterize the behaviour of a flame driven by temperature and pressure variables. The model is formulated using a p-Laplacian operator, an advection term, and a nonlinear reaction (considering linear kinetics). First, the uniqueness and boundedness of the weak solutions are demonstrated. Subsequently, traveling wave solutions supported by the geometric perturbation theory are obtained. As a major outcome, minimum traveling wave speeds are shown to exist, for which the associated profiles of the solutions are purely monotonic with exponential behaviour. The assumptions considered in the analytical approach are further explored through a numerical assessment, and self-similar solutions are constructed to determine the evolution of the flame front in terms of the temperature and pressure variables.

Keywords: p-Laplacian, flame propagation, travelling wave, finite propagation, geometric perturbation theory

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(Some figures may appear in colour only in the online journal)

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1. Introduction

The study of physical and chemical phenomena in spatial domains (including porous media) has attracted interest in different fields, such as chemical kinetics, biology, transport processes, climate change, and combustion (further details on the last two cases can be found in [1, 2], respectively).

Particularly, the characterization of the chemical and physical properties of combustion in porous media is a topic that attracts significant research interest. The combustion process in porous media can be classified into three different types: rapid, spontaneous, and explosive [3–5]. The modelling of the geometry and properties of porous media can be classified into two general concepts. The first concept is related to consolidated and continuous porous media (e.g. the case of non-interconnected regular channels used to support catalytic reactions); whereas the second concept is related to the use of packed discrete elements to build the porous domains (e.g. it can be observed by modelling solid porous sponges in fluid flows with a highly developed surface area at low-pressure drops [6–10]).

The mathematical study of propagating actions in reaction-diffusion models was first introduced by Kolmogorov *et al* [11] and Fisher [12]. The authors obtained analytical solutions to a nonlinear parabolic equation by searching for a special type of solution known as a traveling wave.

It is worth noting that traveling-wave solutions have been employed in several applications involving high-porosity domains. Shchelkin [13] considered the flow of gases passing through a tube and obtained analytical solutions for the shape of a traveling wave. Brailovsky *et al* [14] considered the flow of gases through narrow pipes with large resistances and obtained new solutions based on waves propagating at a constant speed. Subsequently, wave solutions influenced by high-pressure and slow-spreading thermo-diffusion phenomena were reported [15–19]. Gordon [20] characterized the quenching (decay) of solutions in a porous medium for a flow induced under high-pressure conditions and for sufficiently small initial data. Additionally, according to the analysis in [20], if the initial data are sufficiently large, the quenching of the solution occurs when the speed of the propagating wave is of order ϵ_3 (where ϵ_3 refers to the ratio of the specific heat and thermal diffusivity).

The authors used the following governing equations:

$$T_t - \left(1 - \frac{1}{\gamma_3}\right) P_t = \epsilon_3 T_{zz} + Y\Sigma(T), \tag{1.1}$$

$$P_t - T_t = P_{zz}, \tag{1.2}$$

$$Y_t = \epsilon_3 (Le)^{-1} Y_{zz} - \gamma_3 Y\Sigma(T), \tag{1.3}$$

where T is the normalized temperature, P is the pressure, Y is the concentration of defective impulse (or deficient reactant), $\Sigma(T)$ is the normalized reaction rate, Le is the Lewis number, $\gamma_3 > 1$ is the specific heat ratio and ϵ_3 is the ratio of the specific heat and the thermal diffusivity. We remark that the equation (1.1) describes the linear energy conservation law. Equation (1.2) describes the classical law of fluid continuity in combination with the state equation, and equation (1.3) describes the evolution of the flame defective impulse.

Thereafter, Gordon introduced two new variables, u and v, coupled with the physical variables of temperature and pressure as follows [21]

$$T(z,t) = hu(z,t) + (1-h)v(z,t), \; P(z,t) = \frac{1}{\epsilon_3}\Big(u(z,t) + \epsilon_3 v(z,t)\Big) \tag{1.4}$$

where $h = \frac{\mu_1}{1-\epsilon_3} > 0$ and $\mu_1 = \frac{\sqrt{\left[\left(\gamma_3-\epsilon_3\right)^2+4\left(\gamma_3-1\right)\right]}+\epsilon_3-\gamma_3}{2\epsilon_3} > 0$ are the solutions for the following algebraic equation

$$(1-\mu_1)(\gamma_3+\epsilon_3\mu_1) = 1. \tag{1.5}$$

After introducing the scaling $z \to (1-\mu)^{\frac{1}{2}}z$, the system of equations (1.1)–(1.3) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + Y\Sigma(T), \tag{1.6}$$

$$\frac{\partial v}{\partial t} = \epsilon_4\frac{\partial^2 v}{\partial z^2} + Y\Sigma(T), \tag{1.7}$$

$$\frac{\partial Y}{\partial t} = \epsilon_4((1-\mu_1)Le)^{-1}\frac{\partial^2 Y}{\partial z^2} - Y\Sigma(T), \tag{1.8}$$

$$T(z,t) = hu(z,t) + (1-h)v(z,t), \tag{1.9}$$

where $h \in (0,1]$ and $\epsilon_4 = \epsilon_3(1-\mu_1)^2$. Based on the physical observations of combustion and flame propagation, it is generally considered that the ratio of thermal and pressure diffusivities is small. This leads to the assumption that $\epsilon_4 \in (0,1]$ [15]. Now, assuming $Y = 1-v$, equations (1.7) and (1.8) can be shown to be identical by standard operations. In addition, if we consider $Le = Le^* = (1-\mu_1)^{-1}$, this situation resembles the case of regular thermo-diffusive combustion, when the Lewis number equals unity. Further, when $Y = 1-v$, equations (1.6)–(1.9) become

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + (1-v)\Sigma(w), \tag{1.10}$$

$$\frac{\partial v}{\partial t} = \epsilon_4\frac{\partial^2 v}{\partial z^2} + (1-v)\Sigma(w), \tag{1.11}$$

$$w(z,t) = hu(z,t) + (1-h)v(z,t). \tag{1.12}$$

This last set of equations can be expressed in terms of linear kinetics as follows, $\Sigma(w) = w$ [22].

Note that if we set $h = 0$, equation (1.11) describes the classical Kolmogorov-Petrovskii-Piskunov (KPP)-Fisher model [12]. For $h = 1$, the above system was discussed by Billingham and Needham [23] to model unequal diffusion and travelling waves in quadratic and cubic autocatalysis.

As an alternative to the previously discussed models (but strongly influenced by them), we introduce a p-Laplacian operator to enhance the generalization of the diffusive term in equations (1.10)–(1.12), such that it can be applied to a wide range of diffusion driven domains (including porous media). To further justify the use of a p-Laplacian rather than the classical second-order Laplacian, we recall that the shear stresses in a fluid are described by a relationship involving the relative motion between adjacent fluid layers. These relations may be represented by a linear term (leading to the so call Newtonian fluids). However, in general, we can consider that shear stress is expressed as a relation of the form $|\nabla U|^{p-2}$, where U is any open fluid velocity profile and $p \in \mathbb{R}$ is introduced to consider a general form of fluid relative motion. Notably, for $p < 2$, the fluid is typically non-viscous (e.g. volatile gases, super-flowing fluids and unsaturated flows). For $p = 2$, we introduce the case of a linear relation leading to Newtonian fluids (mono-phase water and air flows), whereas for $p > 2$ the flow occurs under relatively high viscosity condition (oils, sand beds or porous medium flow). When considering such a term $|\nabla U|^{p-2}$, for the description of the shear stress effects in a flow and performing standard well documented operations, the equation of fluid momentum can be expressed as a p-Laplacian operator, as considered in our modelling baseline.

To the best of the authors' knowledge, the use of a p-Laplacian operator in flame propagation has not been reported. Nonetheless, this operator has been employed in liquid filtration theory in porous media. Smreker introduced a p-Laplacian term to increase the accuracy of the classical Darcy's law, which is often employed to model flows in porous domains [24, 25]. To further support Smreker's theoretical work, Missbach made relevant contributions to the experimental study [26] and promoted the determination of particular values of p. At this point, we emphasize that our idea is similar to that of Smreker, who first introduced mathematical analyses that can support experimental results related to finding specific values of p. However, we introduce the p-Laplacian operator because of certain physical implications linked with p-Laplacian mathematical properties that aid in modelling flame propagation. We are particularly concerned with the fact that the p-Laplacian preserves energy during propagation and compact support for any given compactly supported initial distribution. Several discussions in this regard, along with applications of the p-Laplacian operator, can be found in the literature [27–30]. In addition, we introduce an advection term to model the possible influences of a backflow in the domain that can affect the evolution of temperature- and pressure-driven flames ([31, 32] presents additional insights into the influences of the advection terms on the flame propagation and burning velocities). Based on the previous discussion, we introduce the p-Laplacian operator and advection term as follows:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{p-2}\frac{\partial u}{\partial z}\right) + c_1\frac{\partial u}{\partial z} + (1-v)w \\
\frac{\partial v}{\partial t} &= \frac{\partial}{\partial z}\left(\left|\frac{\partial v}{\partial z}\right|^{p-2}\frac{\partial v}{\partial z}\right) + c_1\frac{\partial v}{\partial z} + (1-v)w \\
w(z,t) &= hu(z,t) + (1-h)v(z,t),
\end{aligned} \tag{1.13}
$$

where $z\in\mathbb{R}$, $p>2$ (this is typically the case of high resistance propagation effects such as in porous domains), $c_1\in\mathbb{R}$ is the advection coefficient, and $h\in(0,1]$. The initial conditions are positive $u(z,0)=u_0(z)>0$, $v(z,0)=v_0(z)>0$, and comply with a decaying condition at $z\to\infty$, such that $|u_0|$, $|u_0'|$ and $|u_0''|$ (similar for v_0) tend to zero. In our analysis, we take $\epsilon_4=1$ for simplicity and no loss of generality.

To the best of our knowledge, the system defined in equation (1.13) has not been previously considered in the literature. Hence, we first introduce our analysis of the boundedness and uniqueness of weak solutions. Subsequently, we introduce the principles for discussing the stability of the critical point of the system. For this purpose, we converted our system into travelling waves and applied topological concepts in phase space. We applied a singular perturbation procedure, generally known as the geometric perturbation theory, to obtain analytical forms of travelling wave profiles. Further, we complemented these assessments with a numerical approach to examine the consistency of the analytical conception. Eventually, we used self-similar solutions to obtain analytical expressions for the solutions and propagating supports in the flame.

2. Previous results

To prove our results, we first require propositions 1 and 2 to be defined. Proposition 1 is adapted from [16, 27, 28], whereas proposition 2 requires additional proofs. Note that result 1 in [27] was provided for growing functions, and they were not Lebesgue integrable in space. In addition, a norm was provided for measuring the growing condition of the solution trace (this actually provided the conditions to state whether solutions existed globally). The norm was expressed as follows (refer to equation (2.2) in [27]):

$$|||G|||_\rho = \sup_{R \geqslant \rho} R^{-\lambda} \int_{B_R(0)} |G|\,\mathrm{d}z, \quad \lambda = 1 + \frac{p}{p-2}. \tag{2.1}$$

With no loss of generality, and to make our assessments tractable, we assume that $\rho = 1$ in equation (2.1). We remark that given any $\rho > 0$, it holds that $|||G|||_\rho < \infty$ if and only if for certain $\rho_0 > 0$, $|||G|||_{\rho_0} < \infty$ (see p 176 in [16]). If $G \in L^1(\mathbb{R})$, it is easy to verify that $|||G|||_\rho \leqslant ||G||_1$ for $R \geqslant 1$. Based on this, we adjust result 1 of [27] and theorem 2.2.3 in [16] as the following proposition.

Proposition 1. *Assuming that $w_{01}(x) \in L^1(\mathbb{R}^d)$, $d \geqslant 1$, there exist weak solutions for the homogeneous p-Laplacian equation $\frac{\partial w_1}{\partial t} = \nabla \cdot \left(|\nabla w_1|^{p-2} \nabla w_1\right)$ in the waiting time interval $0 < t \leqslant T(w_{01})$ where*

$$T(w_{01}) = E_1(d,p)|||w_{01}|||_1^{-(p-2)}. \tag{2.2}$$

In the finite domain $B_R(0)$, the following relations hold:

$$|||w_1(\cdot,t)|||_1 \leqslant E_2(d,p)|||w_{01}|||_1 \leqslant E_2(d,p)||w_{01}||_1, \tag{2.3}$$

$$|w_1(x,t)| \leqslant E_3(d,p)t^{-a_1}R^{p(p-2)}|||w_{01}|||_1^{\frac{pa_1}{d}} \leqslant E_3(d,p)t^{-a_1}R^{p(p-2)}||w_{01}||_1^{\frac{pa_1}{d}}, \ |x| \leqslant R, \tag{2.4}$$

$$|\nabla w_1(x,t)| \leqslant E_4(d,p)t^{-\frac{(d+1)a_1}{d}}R^{\frac{2}{p-2}}|||w_{01}|||_1^{\frac{2a_1}{d}} \leqslant E_4(d,p)t^{-\frac{(d+1)a_1}{d}}R^{\frac{2}{p-2}}||w_{01}||_1^{\frac{2a_1}{d}}, \ |x| \leqslant R, \tag{2.5}$$

where $a_1 = (p - 2 + \frac{p}{d})^{-1}$ assuming that $|||w_{01}|||_1 \geqslant 1$. Note that $||w_{01}||_1$ was assessed in $B_R(0)$ satisfying $||w_{01}||_{L^1(B_R(0))} \geqslant 1$, $R \geqslant 1$. Additional information regarding constants E_1, E_2, E_3 and E_4 can be found in [27] and theorem 2.2.3 in [16].

The following proposition considers particular forms of the upper bounds for the solutions to equation (1.13). We provide specific forms of the upper and monotonic bounds that complement the general ideas introduced in [27, 28]. Specifically, we attempt to obtain these bounds in the case of predominant advection such that the postulated transport equations can be solved using standard techniques. To prove that the suggested functions are indeed upper bounds to the solutions of equation (1.13), we require additional justifications.

Proposition 2. *The following expressions represent two upper bounds of solutions for the case of predominant advection, compared with the p-Laplacian term, for the system in equation (1.13):*

$$u(z,t) = u_0(z + c_1 t)e^{(h+1)t} + \frac{1}{1+h} e^{-k_1(z+c_1 t)}\left(e^{(h+1)t} - 1\right), \tag{2.6}$$

and

$$v(z,t) = v_0(z + c_1 t)e^{(h+1)t} + \frac{h}{1+h} e^{-k_1(z+c_1 t)}\left(e^{(h+1)t} - 1\right), \tag{2.7}$$

where $k_1 > 0$.

Proof. We aim to determine two bounding functions based on the transport equation of system (1.13). We assume that advection terms are predominant over diffusion-driven terms, $\frac{\partial}{\partial z}(|\frac{\partial u}{\partial z}|^{p-2}\frac{\partial u}{\partial z}) \ll c_1\frac{\partial u}{\partial z}$, $\frac{\partial}{\partial z}(|\frac{\partial v}{\partial z}|^{p-2}\frac{\partial v}{\partial z}) \ll c_1\frac{\partial v}{\partial z}$. Hence, we considered the following equations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= c_1\frac{\partial u}{\partial z} + \Big(1-v\Big)w \\ \frac{\partial v}{\partial t} &= c_1\frac{\partial v}{\partial z} + \Big(1-v\Big)w \\ w &= hu + (1-h)v. \end{aligned} \tag{2.8}$$

Because u, v and w are considered positive solutions, we introduce the following ideas in the search for an upper evolution:

$$\begin{aligned} \frac{\partial u}{\partial t} &= c_1\frac{\partial u}{\partial z} + w - vw \leqslant c_1\frac{\partial u}{\partial z} + w \\ \frac{\partial v}{\partial t} &= c_1\frac{\partial v}{\partial z} + w - vw \leqslant c_1\frac{\partial v}{\partial z} + w \\ w &\leqslant hu + v. \end{aligned} \tag{2.9}$$

By changing the notation, such that u_1 and v_1 represent the functions that provide the upper boundedness of any solution to equation (2.8), we have

$$\frac{\partial u_1}{\partial t} = c_1\frac{\partial u_1}{\partial z} + hu_1 + v_1, \tag{2.10}$$

$$\frac{\partial v_1}{\partial t} = c_1\frac{\partial v_1}{\partial z} + hu_1 + v_1, \tag{2.11}$$

with the same initial conditions as given for the system presented in equation (1.13).

Subtracting equation (2.11) from equation (2.10) and making $f(z,t) = u_1(z,t) - v_1(z,t)$, we obtain

$$\frac{\partial f}{\partial t} = c_1\frac{\partial f}{\partial z}. \tag{2.12}$$

After solving by standard separation of variables, we obtain

$$f(z,t) = e^{-k_1(z+c_1t)}, \tag{2.13}$$

where $k_1 > 0$. Inserting $v_1(z,t) = u_1(z,t) - e^{-k_1(z+c_1t)}$ into equation (2.10) and $u_1(z,t) = v_1(z,t) + e^{-k_1(z+c_1t)}$ into equation (2.11), we obtain

$$\frac{\partial u_1}{\partial t} = c_1\frac{\partial u_1}{\partial z} + (h+1)u_1 - e^{-k_1(z+c_1t)}, \tag{2.14}$$

$$\frac{\partial v_1}{\partial t} = c_1\frac{\partial v_1}{\partial z} + (h+1)v_1 + he^{-k_1(z+c_1t)}. \tag{2.15}$$

To determine the solution to equation (2.15), we consider the following characteristic changes $S = z + c_1t$ and $\tau = t - c_1z$. Subsequently, we obtain

$$(1+c_1^2)\frac{\partial u_1}{\partial \tau} - (h+1)u_1 = -e^{-k_1S}. \tag{2.16}$$

After solving equation (2.16), we obtain

$$u_1(S,\tau) = F(S)e^{\frac{h+1}{1+c_1^2}\tau} + \frac{1}{1+h}e^{-k_1S}, \tag{2.17}$$

which implies that

$$u_1(z,t) = F(z+c_1t)e^{\frac{h+1}{1+c_1^2}(t-c_1z)} + \frac{1}{1+h}e^{-k_1(z+c_1t)}, \tag{2.18}$$

where $F(z+c_1t)$ is an arbitrary function determined using the initial conditions. Substituting the value $F(z+c_1t)$ into equation (2.18), we obtain

$$u_1(z,t) = u_0(z+c_1t)e^{(h+1)t} - \frac{1}{1+h}e^{-k_1(z+c_1t)}\left(e^{(h+1)t}-1\right). \tag{2.19}$$

Similarly, the solution to equation (2.15) is

$$v_1(z,t) = v_0(z+c_1t)e^{(h+1)t} + \frac{h}{1+h}e^{-k_1(z+c_1t)}\left(e^{(h+1)t}-1\right). \tag{2.20}$$

We now prove that equations (2.19) and (2.20) are the upper bounds of any solution to equation (1.13). This can be achieved via direct substitution in equation (1.13) and performing standard operations.

In addition, we show that the hypothesis $\frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{p-2}\frac{\partial u}{\partial z}\right) \ll c_1\frac{\partial u}{\partial z}$, $\frac{\partial}{\partial z}\left(\left|\frac{\partial v}{\partial z}\right|^{p-2}\frac{\partial v}{\partial z}\right) \ll c_1\frac{\partial v}{\partial z}$ is supported. Thus,

$$u_0''(z+c_1t) \leqslant \frac{k_1^2}{1+h}e^{-k_1(z+c_1t)}\left(1-e^{-(h+1)t}\right), \tag{2.21}$$

and

$$v_0''(z+c_1t) \leqslant \frac{k_1^2}{1+h}e^{-k_1(z+c_1t)}\left(2+h-e^{-(h+1)t}\right), \tag{2.22}$$

leading to state that $\frac{\partial}{\partial z}(|\frac{\partial u}{\partial z}|^{p-2}\frac{\partial u}{\partial z}) \ll c_1\frac{\partial u}{\partial z}$ and $\frac{\partial}{\partial z}(|\frac{\partial v}{\partial z}|^{p-2}\frac{\partial v}{\partial z}) \ll c_1\frac{\partial v}{\partial z}$, for any value of c_1 assumed as arbitrarily large to maintain the mentioned orders. $\square$

3. On existence and uniqueness

This section analyses the existence and uniqueness of the solutions to the system of equation (1.13). First, we introduce the following results:

Theorem 1. *Suppose that, if $u(z,t) \geqslant 0$ and $v(z,t) \geqslant 0$ are the weak solutions of (1.13), then these solutions are bounded on $B_r \times (0,T]$, where the ball $B_r = \{z \in \mathbb{R}; |z| \leqslant r\} \in \mathbb{R}$ (it can be identified with the ball $B_R(0)$ in proposition 1; however, we prefer to retain notation B_r for clarity), with $0 < r < \infty$, and $0 < T < \infty$.*

Proof. First, let us consider a test function complying with $\phi_1 \in C^\infty([0,T]\times B_r) \cap L^\infty([0,T], W^{1,p}(B_r))$. Then, for $0 < t \leqslant T$, the weak formulation of equation (1.13) is expressed as:

$$\int_{B_r} u(t)\phi_1(t) = \int_{B_r} u(0)\phi_1(0) + \int_0^t\int_{B_r}\left(u\frac{\partial\phi_1}{\partial s} - \left|\frac{\partial u}{\partial z}\right|^{p-1}\frac{\partial\phi_1}{\partial z} - u\frac{\partial\phi_1}{\partial z} + \left(1-v\right)\phi_1 w\right)\mathrm{d}s, \tag{3.1}$$

$$\int_{B_r} v(t)\phi_1(t) = \int_{B_r} v(0)\phi_1(0) + \int_0^t \int_{B_r} \left(v\frac{\partial \phi_1}{\partial s} - \left|\frac{\partial v}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z} - v\frac{\partial \phi_1}{\partial z} + \left(1-v\right)\phi_1 w \right) \mathrm{d}s. \tag{3.2}$$

To show the boundedness of weak solutions to equation (1.13), we must show that the following evolution problem (denoted as P^{ϕ_1}) is bounded as well:

$$u\frac{\partial \phi_1}{\partial t} - \left|\frac{\partial u}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z} - u\frac{\partial \phi_1}{\partial z} + \left(1-v\right)\phi_1 w = 0, \tag{3.3}$$

$$v\frac{\partial \phi_1}{\partial t} - \left|\frac{\partial v}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z} - v\frac{\partial \phi_1}{\partial z} + \left(1-v\right)\phi_1 w, \tag{3.4}$$

wherein a particular form of a test function is defined as:

$$\phi_1(z,t) = \frac{e^{-k_2(t)}}{(1+|z|^2)^{k_3}}, \tag{3.5}$$

where $k_2(t) \geqslant 0$ is a continuous function and $k_3 > 0$ is a constant that will be specified later in our arguments.

Now, we consider a cut off function (ψ_{λ_1}), with $\lambda_1 \in \mathbb{R}^+$ [33] of the form:

$$\begin{gathered}\psi_{\lambda_1} \in C_0^\infty(z,t), \qquad 0 \leqslant \psi_{\lambda_1} \leqslant 1, \\ \psi_{\lambda_1} = 1 \text{for } z \in B_{r-\lambda_1} \text{and } \psi_{\lambda_1} = 0 \text{for } z \in \{\mathbb{R} - B_{r-\lambda_1}\}.\end{gathered} \tag{3.6}$$

Then, we obtain

$$|\nabla \psi_{\lambda_1}| \leqslant \frac{E_5}{\lambda_1}, \quad |\Delta \psi_{\lambda_1}| \leqslant \frac{E_5}{\lambda_1^2}, \tag{3.7}$$

where $E_5 > 0$. Multiplying equations (3.3) and (3.4) by ψ_{λ_1}, the following hold for $(z,t) \in B_r \times [0,t]$ with $0 < t \leqslant T$:

$$\int_0^t \int_{B_r} u\frac{\partial \phi_1}{\partial t}\psi_{\lambda_1} = \int_0^t \int_{B_r} \left(\left|\frac{\partial u}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z}\psi_{\lambda_1} - u\frac{\partial \phi_1}{\partial z}\psi_{\lambda_1} - \left(1-v\right)\phi_1 w\psi_{\lambda_1} \right) \mathrm{d}s, \tag{3.8}$$

$$\int_0^t \int_{B_r} v\frac{\partial \phi_1}{\partial t}\psi_{\lambda_1} = \int_0^t \int_{B_r} \left(\left|\frac{\partial v}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z}\psi_{\lambda_1} - v\frac{\partial \phi_1}{\partial z}\psi_{\lambda_1} - \left(1-v\right)\phi_1 w\psi_{\lambda_1} \right) \mathrm{d}s, \tag{3.9}$$

where ϕ_1, $\frac{\partial \phi_1}{\partial t}$ and ψ_{λ_1} approach to zero at $z \to \infty$.

Now, our objective is to apply the defined test function (3.5) and propositions 1 and 2 to exhibit the boundedness of the integrals $\int_0^t \int_{B_r} u\frac{\partial \phi_1}{\partial t}\psi_{\lambda_1}$ and $\int_0^t \int_{B_r} v\frac{\partial \phi_1}{\partial t}\psi_{\lambda_1}$. The integrals involved in equations (3.8) and (3.9) can be evaluated as

$$\begin{aligned}\int_0^t \int_{B_r} \left|\frac{\partial u}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z}\psi_{\lambda_1} \mathrm{d}s \leqslant \int_0^t \int_{B_r} \Bigg(& E_4^{p-1} s^{-2a_1(p-1)} |z|^{2+\frac{2}{p-2}} \|u_0\|_1^{2a_1(p-1)} \\ & + \left(u_0(z+c_1 s)e^{(h+1)s} + \frac{1}{1+h} e^{-k_1(z+c_1 s)} \left(e^{(h+1)s} - 1 \right) \right)^{p-1} \Bigg) \frac{2|-k_3||z|e^{-k_2(s)}}{(1+|z|^2)^{k_3+1}} \psi_{\lambda_1} \mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^t \int_{B_r} \Big(E_4^{p-1} s^{-2a_1(p-1)} |z|^{1+\frac{2}{p-2}-2k_3} \|u_0\|_1^{2a_1(p-1)} \\
&\quad + \Big(u_0(z+c_1 s)e^{(h+1)s} + \frac{1}{1+h} e^{-k_1(z+c_1 s)} \Big(e^{(h+1)s}-1\Big)\Big)^{p-1} z^{-1-2k_3}\Big) 2k_3 e^{-k_2(s)} \psi_{\lambda_1} \mathrm{d}s \\
&\leqslant \int_0^t \int_{B_r} \Big(E_4^{p-1} s^{-2a_1(p-1)} |z|^{1+\frac{2}{p-2}-2k_3} \|u_0\|_1^{2a_1(p-1)} \\
&\quad + \Big(u_0(z+c_1 s)e^{(h+1)s} + \frac{1}{1+h} e^{-k_1(z+c_1 s)} \Big(e^{(h+1)s}-1\Big)\Big)^{p-1} z^{-1-2k_3}\Big) 2k_3 e^{-k_2(s)} \frac{E_5}{\lambda_1} \mathrm{d}s \\
&\leqslant \int_0^t \int_{B_r} \Big(E_4^{p-1} s^{-2a_1(p-1)} |z|^{\frac{2}{p-2}-2k_3} \|u_0\|_1^{2a_1(p-1)} \\
&\quad + \Big(u_0(z+c_1 s)e^{(h+1)s} + \frac{1}{1+h} e^{-k_1(z+c_1 s)} \Big(e^{(h+1)s}-1\Big)\Big)^{p-1} z^{-2-2k_3}\Big) 2E_5 k_3 e^{-k_2(t)} \mathrm{d}s. \qquad (3.10)
\end{aligned}
$$

Considering $k_3 \geqslant \frac{1}{p-2} = k_4$, the above expression converges asymptotically. With asymptotic convergence, we find that, for $z \gg 1$ ($r \gg 1$) and finite, the integral approaches zero.

Similarly

$$
\begin{aligned}
&\int_0^t \int_{B_r} \left|\frac{\partial v}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z} \psi_{\lambda_1} \mathrm{d}s \leqslant \int_0^t \int_{B_r} \Big(E_4^{p-1} s^{-2a_1(p-1)} |z|^{\frac{2}{p-2}-2k_3} \|u_0\|_1^{2a_1(p-1)} \\
&\quad + \Big(v_0(z+c_1 s)e^{(h+1)s} - \frac{h}{1+h} e^{-k_1(z+c_1 s)} \Big(e^{(h+1)s}-1\Big)\Big)^{p-1} z^{-2-2k_3}\Big) 2E_5 k_3 e^{-k_2(s)} \mathrm{d}s, \qquad (3.11)
\end{aligned}
$$

which is asymptotically convergent if $k_3 \geqslant \frac{1}{p-2} = k_5$. The second integral on the right hand side of equation (3.8) is assessed as

$$
\begin{aligned}
&-\int_0^t \int_{B_r} u \frac{\partial \phi_1}{\partial z} \psi_{\lambda_1} \mathrm{d}s \leqslant \int_0^t \int_{B_r} \Big(E_3 s^{-a_1} |z|^{p(p-2)} \|u_0\|_1^{pa_1} + u_0(z+c_1 s) e^{(h+1)s} \\
&\quad + \frac{1}{1+h} e^{-k_1(z+c_1 s)} \Big(e^{(h+1)s}-1\Big)\Big) \frac{2|-k_3||z| e^{-k_2(s)}}{(1+|z|^2)^{k_3+1}} \psi_{\lambda_1} \mathrm{d}s \\
&\leqslant \int_0^t \int_{B_r} \Big(E_3 s^{-a_1} |z|^{p(p-2)-2k_3-1} \|u_0\|_1^{pa_1} + u_0(z+c_1 s) e^{(h+1)s} \\
&\quad + \frac{1}{1+h} e^{-k_1(z+c_1 s)} \Big(e^{(h+1)s}-1\Big) z^{-1-2k_3}\Big) 2k_3 e^{-k_2(s)} \psi_{\lambda_1} \mathrm{d}s \\
&\leqslant \int_0^t \int_{B_r} \Big(E_3 s^{-a_1} |z|^{p(p-2)-2k_3-1} \|u_0\|_1^{pa_1} + u_0(z+c_1 s) e^{(h+1)s} \\
&\quad + \frac{1}{1+h} e^{-k_1(z+c_1 s)} \Big(e^{(h+1)s}-1\Big) z^{-1-2k_3}\Big) 2k_3 e^{-k_2(s)} \frac{E_5}{\lambda_1} \mathrm{d}s \\
&\leqslant \int_0^t \int_{B_r} \Big(E_3 s^{-a_1} |z|^{p(p-2)-2k_3-2} \|u_0\|_1^{pa_1} + u_0(z+c_1 s) e^{(h+1)s} \\
&\quad + \frac{1}{1+h} e^{-k_1(z+c_1 s)} \Big(e^{(h+1)s}-1\Big) z^{-2-2k_3}\Big) 2E_5 k_3 e^{-k_2(t)} \mathrm{d}s. \qquad (3.12)
\end{aligned}
$$

With the choice of $k_3 \geqslant -1 = k_6$, the above integral converges asymptotically.

Similar to the previous case,

$$-\int_0^t\int_{B_r} v\frac{\partial\phi_1}{\partial z}\psi_{\lambda_1}\mathrm{d}s \leqslant \int_0^t\int_{B_r}\Big(E_3 s^{-a_1}|z|^{p(p-2)-2k_3-2}\left\|v_0\right\|_1^{pa_1}$$
$$+v_0(z+c_1s)e^{(h+1)s}+\frac{1}{1+h}e^{-k_1(z+c_1s)}\Big(e^{(h+1)s}-1\Big)z^{-2-2k_3}\Big)2E_5k_3e^{-k_2(t)}\mathrm{d}s, \tag{3.13}$$

which converges asymptotically if $k_3 \geqslant -1 = k_6.$

An assessment of the last integrals in equations (3.8) and (3.9) is expressed as follows:

$$\begin{aligned}
&\int_0^t\int_{B_r}\Big(1-v)\Big)w\phi_1\psi_{\lambda_1}\mathrm{d}s \leqslant \int_0^t\int_{B_r} w\phi_1\psi_{\lambda_1}\mathrm{d}s\\
&\leqslant h\int_0^t\int_{B_r} u\phi_1\psi_{\lambda_1}\mathrm{d}s + \int_0^t\int_{B_r} v\phi_1\psi_{\lambda_1}\mathrm{d}s\\
&\leqslant h\int_0^t\int_{B_r}\Big(E_3s^{-a_1}|z|^{p(p-2)}\left\|u_0\right\|_1^{pa_1}+u_0(z+c_1s)e^{(h+1)s}\\
&\quad+\frac{1}{1+h}e^{-k_1(z+c_1s)}\Big(e^{(h+1)s}-1\Big)\Big)\frac{e^{-k_2(s)}}{(1+|z|^2)^{k_3}}\psi_{\lambda_1}\mathrm{d}s\\
&\quad+\int_0^t\int_{B_r}\Big(E_3s^{-a_1}|z|^{p(p-2)}\left\|v_0\right\|_1^{pa_1}+v_0(z+c_1s)e^{(h+1)s}\\
&\quad+\frac{h}{1+h}e^{-k_1(z+c_1s)}\Big(e^{(h+1)s}-1\Big)\Big)\frac{e^{-k_2(s)}}{(1+|z|^2)^{k_3}}\psi_{\lambda_1}\mathrm{d}s\\
&\leqslant h\int_0^t\int_{B_r}\Big(E_3s^{-a_1}|z|^{p(p-2)-2k_3}\left\|u_0\right\|_1^{pa_1}+u_0(z+c_1s)e^{(h+1)s}\\
&\quad+\frac{1}{1+h}e^{-k_1(z+c_1s)}\Big(e^{(h+1)s}-1\Big)z^{-2k_3}\Big)e^{-k_2(s)}\psi_{\lambda_1}\mathrm{d}s\\
&\quad+\int_0^t\int_{B_r}\Big(E_3s^{-a_1}|z|^{p(p-2)-2k_3}\left\|v_0\right\|_1^{pa_1}+v_0(z+c_1s)e^{(h+1)s}\\
&\quad+\frac{1}{1+h}e^{-k_1(z+c_1s)}\Big(e^{(h+1)s}-1\Big)z^{-2k_3}\Big)e^{-k_2(s)}\psi_{\lambda_1}\mathrm{d}s\\
&\leqslant h\int_0^t\int_{B_r}\Big(E_3s^{-a_1}|z|^{p(p-2)-2k_3}\left\|u_0\right\|_1^{pa_1}+u_0(z+c_1s)e^{(h+1)s}\\
&\quad+\frac{h}{1+h}e^{-k_1(z+c_1s)}\Big(e^{(h+1)s}-1\Big)z^{-2k_3}\Big)e^{-k_2(s)}\frac{E_5}{\lambda_1}\mathrm{d}s\\
&\quad+\int_0^t\int_{B_r}\Big(E_3s^{-a_1}|z|^{p(p-2)-2k_3}\left\|v_0\right\|_1^{pa_1}+v_0(z+c_1s)e^{(h+1)s}\\
&\quad+\frac{h}{1+h}e^{-k_1(z+c_1s)}\Big(e^{(h+1)s}-1\Big)z^{-2k_3}\Big)e^{-k_2(s)}\frac{E_5}{\lambda_1}\mathrm{d}s.
\end{aligned} \tag{3.14}$$

The integrals are asymptotically convergent in equation (3.14) if $k_3 \geqslant -\frac{1}{2} = k_7$.

In conclusion, we state that the integrals given in the set of equations (3.10)–(3.14) converge if we select

$$k_3 \geqslant \max\left(k_4, k_5, k_6, k_7\right). \tag{3.15}$$

Combining equations (3.8)–(3.15) while rendering $r \gg 1$ and finite, we obtain

$$0 \leqslant \int_0^t \int_{B_{r\to\infty}} u \frac{\partial \phi_1}{\partial s} \psi_{\lambda_1} \leqslant M_u^+, \tag{3.16}$$

$$0 \leqslant \int_0^t \int_{B_{r\to\infty}} v \frac{\partial \phi_1}{\partial s} \psi_{\lambda_1} \leqslant M_v^+, \tag{3.17}$$

where M_u^+ and M_v^+ are two positive constants that exist given the asymptotic convergence for each integral. Because the test function ϕ_1 is bounded, and after substituting equations (3.16) and (3.17) into (3.1) and (3.2) respectively, we conclude that the solutions u and v are bounded in $(0,T] \times B_r$. □

The next goal is to show that any solution to equation (1.13) is unique. For this purpose, we consider an approach based on the coincidence of a pair of upper and lower solutions. The concepts of the upper and lower solutions used in the following theorem are based on the classical text [34].

Theorem 2. *Assuming that there exists a pair of positive lower solutions for equation (1.13) as $(u(z,t), v(z,t)) \geqslant 0$, this pair of solutions coincides with that of the upper solutions $(\bar{u}(z,t), \bar{v}(z,t))$ in $B_r \times (0,T]$, where $r \gg 1$, $B_r \in \mathbb{R}$ is the classical ball and $0 < T < \infty$. This implies that any pair of solutions to equation (1.13) in $B_r \times (0,T]$ is unique.*

Proof. A pair of upper solutions to equation (1.13), $\bar{u}$ and $\bar{v}$, is constructed with a parameter $\epsilon > 0$ such that the upper solutions depart from the following initial data as follows

$$\bar{u}(z,0) = u_0(z) + \epsilon, \;\; \bar{v}(z,0) = v_0(z) + \epsilon. \tag{3.18}$$

The construction of the upper solutions can be justified based on the monotonic properties of the p-Laplacian operator. Additionally, the following equations satisfy the initial conditions:

$$\frac{\partial}{\partial z}\left(\left| \frac{\partial u_0}{\partial z} \right|^{p-2} \frac{\partial u_0}{\partial z} \right) + c_1 \frac{\partial u_0}{\partial z} + (1 - v_0 - \epsilon)\Big(hu_0 + \big(1-h\big)v_0 + \epsilon\Big) = 0, \tag{3.19}$$

$$\frac{\partial}{\partial z}\left(\left| \frac{\partial v_0}{\partial z} \right|^{p-2} \frac{\partial v_0}{\partial z} \right) + c_1 \frac{\partial v_0}{\partial z} + (1 - v_0 - \epsilon)\Big(hu_0 + \big(1-h\big)v_0 + \epsilon\Big) = 0. \tag{3.20}$$

As mentioned previously, $|u_0| \to 0^+$, $|u_0'| \to 0^+$ and $|u_0''| \to 0^+$ (similar situation for v_0) for $z \gg 1$. Then, considering only the leading terms, the following condition is obtained for the selection of an adequate $\epsilon > 0$:

$$\epsilon = \max\{1 + \|v_0\|_\infty, h\|u_0\|_\infty + (1+h)\|v_0\|_\infty\}. \tag{3.21}$$

Note that u and v refer to the lower solutions of equation (1.13) with

$$u(z,0) = u_0(z), \;\; v(z,0) = v_0(z). \tag{3.22}$$

This is based on the assumption that

$$\phi_1(x,t) = \frac{e^{-k_8 s}}{(1+|z|^2)^{k_3}}, \tag{3.23}$$

where $\phi_1 \in C^\infty \times [0,T]$ is the particular form of a test function. Further, the following assessments hold

$$\int_{B_r} \bar{u}(t)\phi_1(t) = \int_{B_r} \bar{u}(0)\phi_1(0) + \int_0^t \int_{B_r} \left(\bar{u}\frac{\partial \phi_1}{\partial s} - \left|\frac{\partial \bar{u}}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z} - c_1\bar{u}\frac{\partial \phi_1}{\partial z} + (1-\bar{v})\phi_1\bar{w}\right)\mathrm{d}s, \tag{3.24}$$

$$\int_{B_r} \bar{v}(t)\phi_1(t) = \int_{B_r} \bar{v}(0)\phi_1(0) + \int_0^t \int_{B_r} \left(\bar{v}\frac{\partial \phi_1}{\partial s} - \left|\frac{\partial \bar{v}}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z} - c_1\bar{v}\frac{\partial \phi_1}{\partial z} + (1-\bar{v})\phi_1\bar{w}\right)\mathrm{d}s, \tag{3.25}$$

$$\int_{B_r} u(t)\phi_1(t) = \int_{B_r} u(0)\phi_1(0) + \int_0^t \int_{B_r} \left(u\frac{\partial \phi_1}{\partial s} - \left|\frac{\partial u}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z} - c_1\bar{u}\frac{\partial \phi_1}{\partial z} + (1-v)\phi_1 w\right)\mathrm{d}s, \tag{3.26}$$

$$\int_{B_r} v(t)\phi_1(t) = \int_{B_r} v(0)\phi_1(0) + \int_0^t \int_{B_r} \left(v\frac{\partial \phi_1}{\partial s} - \left|\frac{\partial v}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z} - c_1 v\frac{\partial \phi_1}{\partial z} + (1-v)\phi_1 w\right)\mathrm{d}s. \tag{3.27}$$

Similar to theorem 1, the following assessments hold based on a procedure similar to that discussed previously:

$$\begin{gathered}
\int_0^t \int_{B_r} \left|\frac{\partial \bar{u}}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z}\mathrm{d}s = C_1, \quad \int_0^t \int_{B_r} \left|\frac{\partial \bar{v}}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z}\mathrm{d}s = C_2, \quad \int_0^t \int_{B_r} \left|\frac{\partial u}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z}\mathrm{d}s = C_3, \\
\int_0^t \int_{B_r} \left|\frac{\partial v}{\partial z}\right|^{p-1} \frac{\partial \phi_1}{\partial z}\mathrm{d}s = C_4, \quad \int_0^t \int_{B_r} \bar{u}\frac{\partial \phi_1}{\partial z}\mathrm{d}s = C_5 \quad \int_0^t \int_{B_r} \bar{v}\frac{\partial \phi_1}{\partial z}\mathrm{d}s = C_6 \\
\int_0^t \int_{B_r} u\frac{\partial \phi_1}{\partial z}\mathrm{d}s = C_7 \quad \int_0^t \int_{B_r} v\frac{\partial \phi_1}{\partial z}\mathrm{d}s = C_8,
\end{gathered} \tag{3.28}$$

where $C_i;\ i = 1,2,\ldots 8$ are constants. In addition

$$\int_0^t \int_{B_r} \left(1-\bar{v}\right)\bar{w}\phi_1\mathrm{d}s \leqslant \int_0^t \int_{B_r} \bar{w}\phi_1\mathrm{d}s \leqslant h\int_0^t \int_{B_r} \bar{u}\phi_1\mathrm{d}s + \int_0^t \int_{B_r} \bar{v}\phi_1\mathrm{d}s, \tag{3.29}$$

$$\int_0^t \int_{B_r} \left(1-v\right)w\phi_1\mathrm{d}s \leqslant \int_0^t \int_{B_r} w\phi_1\mathrm{d}s \leqslant h\int_0^t \int_{B_r} u\phi_1\mathrm{d}s + \int_0^t \int_{B_r} v\phi_1\mathrm{d}s. \tag{3.30}$$

Using equations (3.28)–(3.30), equations (3.24)–(3.27) become

$$\int_{B_r} \bar{u}(t)\phi_1(t) \leqslant \int_{B_r} \bar{u}(0)\phi_1(0) + \int_0^t \int_{B_r} \bar{u}\frac{\partial \phi_1}{\partial s}\mathrm{d}s + h\int_0^t \int_{B_r} \bar{u}\phi_1 \mathrm{d}s + \int_0^t \int_{B_r} \bar{v}\phi_1 \mathrm{d}s + C_9, \tag{3.31}$$

$$\int_{B_r} \bar{v}(t)\phi_1(t) \leqslant \int_{B_r} \bar{v}(0)\phi_1(0) + \int_0^t \int_{B_r} \bar{v}\frac{\partial \phi_1}{\partial s}\mathrm{d}s + h\int_0^t \int_{B_r} \bar{u}\phi_1 \mathrm{d}s + \int_0^t \int_{B_r} \bar{v}\phi_1 \mathrm{d}s + C_{10}, \tag{3.32}$$

where $C_9 = C_1 + C_5$ and $C_{10} = C_2 + C_6$,

$$\int_{B_r} u(t)\phi_1(t) \leqslant \int_{B_r} u(0)\phi_1(0) + \int_0^t \int_{B_r} u\frac{\partial \phi_1}{\partial s}\mathrm{d}s + h\int_0^t \int_{B_r} u\phi_1 \mathrm{d}s + \int_0^t \int_{B_r} v\phi_1 \mathrm{d}s + C_{11}, \tag{3.33}$$

$$\int_{B_r} v(t)\phi_1(t) \leqslant \int_{B_r} v(0)\phi_1(0) + \int_0^t \int_{B_r} v\frac{\partial \phi_1}{\partial s}\mathrm{d}s + h\int_0^t \int_{B_r} u\phi_1 \mathrm{d}s + \int_0^t \int_{B_r} v\phi_1 \mathrm{d}s + C_{12}, \tag{3.34}$$

where $C_{11} = C_3 + C_7$ and $C_{12} = C_4 + C_8$. On subtracting equation (3.33) from equation (3.31) and equation (3.34) from equation (3.32) and further using equation (3.23) with $\epsilon \to 0$, we obtain

$$\begin{aligned}\int_{B_r} \Big(\bar{u}(t) - u(t)\Big)\phi_1(t) \leqslant -k_8 \int_0^t \int_{B_r} \Big(\bar{u} - u\Big)\phi_1 \mathrm{d}s + h\int_0^t \int_{B_r} \Big(\bar{u} - u\Big)\phi_1 \mathrm{d}s \\ + \int_0^t \int_{B_r} \Big(\bar{v} - v\Big)\phi_1 \mathrm{d}s,\end{aligned} \tag{3.35}$$

$$\begin{aligned}\int_{B_r} \Big(\bar{v}(t) - v(t)\Big)\phi_1(t) \leqslant -k_8 \int_0^t \int_{B_r} \Big(\bar{v} - v\Big)\phi_1 \mathrm{d}s + h\int_0^t \int_{B_r} \Big(\bar{u} - u\Big)\phi_1 \mathrm{d}s \\ + \int_0^t \int_{B_r} \Big(\bar{v} - v\Big)\phi_1 \mathrm{d}s.\end{aligned} \tag{3.36}$$

Given that $k_8 > 0$, the following holds

$$-k_8\Big(\bar{u} - u\Big)\phi_1 \leqslant 0 \text{ and} - k_8\Big(\bar{v} - v\Big)\phi_1 \leqslant 0. \tag{3.37}$$

On differentiating equations (3.35) and (3.36) with respect to t and then subtracting, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{B_r} \Big(\bar{u}(t) - u(t)\Big)\phi_1(t) - \int_{B_r} \Big(\bar{v}(t) - v(t)\Big)\phi_1(t)\right) \leqslant 0, \tag{3.38}$$

where we used equation (3.37). Consequently, applying the Gronwall inequality, we obtain

$$\begin{aligned}&\int_{B_r} \Big(\bar{u}(t) - u(t)\Big)\phi_1(t) - \int_{B_r} \Big(\bar{v}(t) - v(t)\Big)\phi_1(t) \\ &\quad \leqslant \int_{B_r} \Big(\bar{u}(0) - u(0)\Big)\phi_1(0) - \int_{B_r} \Big(\bar{v}(0) - v(0)\Big)\phi_1(0).\end{aligned} \tag{3.39}$$

After using the initial conditions expressed in equations (3.18) and (3.22), the following assessment holds:

$$\int_{B_r}\left(\bar{u}(t)-u(t)\right)\phi_1(t)-\int_{B_r}\left(\bar{v}(t)-v(t)\right)\phi_1(t)\leqslant 2\epsilon, \tag{3.40}$$

where ϵ can be considered as sufficiently small such that

$$\bar{u}(t)-u(t)\sim\bar{v}(t)-v(t)\to 0. \tag{3.41}$$

This implies that $\bar{u}\to u,\ \bar{v}\to v$, thus confirming the asymptotic uniqueness of any solution to equation (1.13).

□

4. Travelling waves analysis

The theory of travelling waves is used in this section to obtain the profiles of the solutions for the problem in equation (1.13). For this objective, we consider the changes $u(z,t)=F_1(\eta)$ and $v(z,t)=F_2(\eta)$ with $\eta=z-c_2t\in(0,\infty)$, where c_2 is the speed of propagation, which is considered common for both forms of travelling wave solutions. The above mentioned travelling wave profiles are provided as functions $F_1:(0,\infty)\to(0,\infty)$ and $F_2:(0,\infty)\to(0,\infty)$. Considering the boundedness condition shown in theorem 1 (no blow-up is expected) we can generally assume that $F_1\in L^\infty(0,\infty)$ and $F_2\in L^\infty(0,\infty)$. Although additional regularity conditions may be added to the mentioned profiles, we prefer to maintain certain minimum requirements and provide further evidence based on subsequent numerical assessments.

In addition, we should recall that two travelling waves are symmetrically equivalent, and this is upon replacement of $(\eta,c_2)\to(-\eta,-c_2)$ and equivalent under translation $\eta\to\eta+\eta_0$, being $\eta_0\in\mathbb{R}$.

System (1.13) is now transformed by considering the exposed changes under the travelling waves formulation, which yields

$$-\left(c_1+c_2\right)F_1'=(p-1)\left(F_1'\right)^{p-2}F_1''+\left(1-F_2\right)F_3, \tag{4.1}$$

and

$$-\left(c_1+c_2\right)F_2'=(p-1)\left(F_2'\right)^{p-2}F_2''+\left(1-F_2\right)F_3, \tag{4.2}$$

where $F_1,F_2,F_3\in L^\infty(0,\infty)$.

Before proceeding further, we introduce a brief summary of the results to be obtained: the system of equations (4.1) and (4.2) is analysed in a phase space considering new variables that will aid in the study of the system critical point. We aim to prove that, in the proximity of the critical point, there exists an exponential behaviour of stable flows and that a minimum travelling wave speed is necessary for a purely monotone condition in the profile (the purpose of lemma 1). Subsequently, we used geometric perturbation theory to support the construction of reduced-order manifolds, for which the computation of analytical profiles of solutions is made easier (and linked with searching for a regular trajectory in the reduced phase plane). Finally, a numerical assessment was conducted to verify the consistency of the analytical approach.

To examine the solutions in a phase space, we introduce density and flux functions, which are defined as

$$X_3=F_1(\zeta),\quad Y_3=\left(F_1'(\zeta)\right)^{p-1},\quad X_4=F_2(\zeta),\quad Y_4=\left(F_2'(\zeta)\right)^{p-1}, \tag{4.3}$$

such that the phase space is given by the dimensions (X_3,Y_3,X_4,Y_4).

Equations (4.1) and (4.2) are expressed as follows

$$
\begin{aligned}
X_3' &= Y_3^{\frac{1}{p-1}}, \\
Y_3' &= -\left(c_1+c_2\right)Y_3^{\frac{1}{p-1}} - \left(1-X_4\right)\left(hX_3+(1-h)X_4\right), \\
X_4' &= Y_4^{\frac{1}{p-1}}, \\
Y_4' &= -\left(c_1+c_2\right)Y_4^{\frac{1}{p-1}} - \left(1-X_4\right)\left(hX_3+(1-h)X_4\right).
\end{aligned}
\tag{4.4}
$$

From the above expressions, a critical subspace is expressed as $(X_3, Y_3, X_4, Y_4) = (m_2, 0, 1, 0)$, where $m_2 \in \mathbb{R}$ is regarded as a free parameter.

Based on the equation (4.4) and the obtained critical subspace, the following lemma holds:

Lemma 1. *For any value of m_2, the critical point $(m_2, 0, 1, 0)$ represents a degenerate node, such that:*

- *There exist three negative and one null eigenvalues, when $h(1-m_2) \leqslant 1$.*
- *There exist two negative, one positive and one null eigenvalues when $h(1-m_2) > 1$.*

In the proximity of the critical point, there exist stable travelling wave profiles of solutions.

Proof. In the proximity of the stationary condition, assume that:

$$\bar{X}_4 = 1 - X_4, \quad X_4 \ll 1. \tag{4.5}$$

Consequently, in such proximity, equation (4.4) can be made linear and expressed as

$$
\begin{pmatrix} X_3' \\ Y_3' \\ \bar{X}_4{}' \\ Y_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\left(c_1+c_2\right) & h(1-m_2)-1 & 0 \\ 0 & 0 & 0 & m_2 \\ 0 & 0 & h(1-m_2)-1 & -\left(c_1+c_2\right) \end{pmatrix} \begin{pmatrix} X_3 \\ Y_3 \\ \bar{X}_4 \\ Y_4 \end{pmatrix}. \tag{4.6}
$$

The eigenvalues of the matrix in equation (4.6) are $\lambda_1 = 0$, $\lambda_2 = -\left(c_1+c_2\right)$, $\lambda_3 = -\frac{c_1+c_2}{2} + \frac{1}{2}\sqrt{\left(c_1+c_2\right)^2 + 4\left[h(1-m_2)-1\right]}$ and $\lambda_4 = -\frac{c_1+c_2}{2} - \frac{1}{2}\sqrt{\left(c_1+c_2\right)^2 + 4\left[h(1-m_2)-1\right]}$.

We can observe that the non-negative eigenvalues exist provided that

$$m_2 \leqslant \frac{1}{h}\left(\frac{\left(c_1+c_2\right)^2}{4} + h - 1\right). \tag{4.7}$$

Furthermore, we state that there exists a minimum travelling wave speed ($c_{2,\min}$), such that for $c_2 \geqslant c_{2,\min}$, the profiles of travelling waves are monotone decreasing or increasing (no oscillations occur), which can be obtained by direct computation in the expression (4.7) as:

$$c_{2,\min} = 2\sqrt{1-h(1-m_2)} - c_1. \tag{4.8}$$

Note that the computed eigenvalues directly lead to the cases stated in lemma 1. The flow associated to the null eigenvalue is $X_3 = m_2$, $\bar{X}_4 = 0$, which by translation implies $X_3 = m_2$, $X_4 = 1$.

Note that in the proximity of $(X_3, Y_3, X_4, Y_4) = (m_2, 0, 1, 0)$, the solutions for the set of equations in equation (4.6) are obtained in the form of exponential profiles as:

$$X_3 = C_1 e^{\lambda t}; \quad X_4 = C_2 e^{\lambda t}, \tag{4.9}$$

indicating the regular stability of the travelling wave solutions near the critical point. Here $C_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, and λ refers to each form of the eigenvalues obtained for the matrix in equation (4.6).

□

4.1. Geometric perturbation theory

In this section, we use the singular perturbation theory, which allows us to define a perturbed manifold linked with equation (4.4). Subsequently, the travelling wave solutions were computed.

First, assume a dimensionally reduced manifold Ω_3 which is expressed as

$$\Omega_3 = \left\{ X_3' = Y_3^{\frac{1}{p-1}}; Y_3' = -\left(c_1 + c_2\right) Y_3^{\frac{1}{p-1}} \right\}, \tag{4.10}$$

with the critical point $(m_2, 0, 1, 0)$ as previously discussed for equation (4.4).

Now, consider the perturbed manifold $\Omega_{3,\epsilon}$, in the proximity of Ω_3. It is defined as

$$\Omega_{3,\epsilon} = \left\{ X_3 = m_2 - \epsilon; X_4 = 1 - \epsilon; X_3' = Y_3^{\frac{1}{p-1}}; Y_3' = -\left(c_1 + c_2\right) Y_3^{\frac{1}{p-1}} - \left[hX_3 + (1-h)X_4 \right] \epsilon \right\}, \tag{4.11}$$

where ϵ_3 is small and is the perturbation parameter near the critical point $(m_2, 0, 1, 0)$.

We establish the hyperbolic condition of perturbed manifold Ω_3 by employing the Fenichel invariant manifold theorem [35–37]. Consequently, we are in a position to obtain the analytical profiles of solutions using the newly defined perturbed manifold $\Omega_{3,\epsilon}$. To achieve this objective, equations (4.1) and (4.2) shall be first linearized near the critical point. Thereafter, it must be shown that the eigenfunctions (related to the linearized eigenvalues with nonzero real part) are transversal to the tangent space of $\Omega_{3,\epsilon}$.

Let us consider the following two-dimensional (2D) flow, which is related to Ω_3 as follows:

$$\begin{pmatrix} X_3' \\ Y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -\left(c_1 + c_2\right) \end{pmatrix} \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}, \tag{4.12}$$

where in the proximity of $Y_3 = 0$, we have considered that $Y_3^{\frac{1}{p-2}} \sim Y_3$. Even when the order of the approach is different, a perturbation analysis is applied in the same manner, as both functions $Y_3^{\frac{1}{p-2}}, Y_3$ tend to zero whenever $Y_3 \to 0^+$.

After solving the above expression, we first obtain eigenvalues 0 and $-(c_1 + c_2)$. Using the null eigenvalue, we obtain $(1, 0)$ as eigenvector, which is tangent to the manifold Ω_3. Further, by the Fenichel invariant theorem, this is equivalent to a state wherein Ω_3 is a hyperbolic manifold.

We intend to use an idea from [36] to examine the local invariant condition of $\Omega_{3,\epsilon}$. Thus, we consider the set

$$\begin{aligned} \psi_1 &= m_2 - \epsilon \\ \psi_2 &= 1 - \epsilon \\ \psi_3 &= -\left(c_1 + c_2\right) Y_3^{\frac{1}{p-1}} \\ \psi_4 &= -\left(c_1 + c_2\right) Y_3^{\frac{1}{p-1}} - \left[hX_3 + (1-h)X_4 \right] \epsilon. \end{aligned} \tag{4.13}$$

Based on [36], for all $r_2 > 0$ and for any value of $j \in N$, there exists a δ_1, such that if $\epsilon \in (0, \delta_1)$, and $j > 0$, functions ψ_i, $i = 1,2,3,4$, in equation (4.13) are $C^j(\bar{B}_{r_2}(0) \times (0, \delta_1))$ in the proximity of the equilibrium $(m_2, 0, 1, 0)$.

We can choose $r_2 > 0$ such that $\Omega_3 \cap B_{r_2}(0)$ is sufficiently large to cover the domain in which the travelling wave evolve. Note that the length between the manifold Ω_3 and its associated perturbed manifold is δ_1. Additionally, functions involved in equation (4.13) are measurable in $B_{r_2}(0)$. Then, the following assessment holds

$$\left\| \psi_4^{\Omega_3} - \psi_4^{\Omega_{3,\epsilon}} \right\| \leqslant \delta_1 \epsilon, \tag{4.14}$$

where $\delta_1 = |hm_2 + (1-h)|$, which implies that for any $\delta_1 \in (0, \infty)$, the length between the manifolds satisfies the normal hyperbolic condition.

To continue with our assessment, we define the manifold Ω_4, in a similar way as that with Ω_3 :

$$\Omega_4 = \left\{ X_4' = Y_4^{\frac{1}{p-1}}; Y_4' = -\left(c_1 + c_2\right) Y_4^{\frac{1}{p-1}} \right\}, \tag{4.15}$$

in proximity to the critical point $(m_2, 0, 1, 0)$, the perturbed manifold $\Omega_{4,\epsilon}$, associated with Ω_4, is defined as

$$\Omega_{4,\epsilon} = \left\{ X_3 = m_2 - \epsilon; X_4 = 1 - \epsilon; X_4' = Y_2^{\frac{1}{p-1}}; Y_4' = -\left(c_1 + c_2\right) Y_4^{\frac{1}{p-1}} - \left[hX_3 + (1-h)X_4 \right] \epsilon \right\}. \tag{4.16}$$

Now, we apply the Fenichel invariant manifold theorem, similar to that in the case of Ω_3. Assuming a 2D flow associated with Ω_4, it is defined as

$$\begin{pmatrix} X_4' \\ Y_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -\left(c_1 + c_2\right) \end{pmatrix} \begin{pmatrix} X_4 \\ Y_4 \end{pmatrix}, \tag{4.17}$$

where in the proximity of $Y_4 = 0$, we have considered that $Y_4^{\frac{1}{p-2}} \sim Y_4$. The perturbation analysis is applicable in the same manner as both functions $Y_4^{\frac{1}{p-2}}, Y_4$ tend to zero whenever $Y_4 \to 0^+$.

The above expression has the eigenvalues 0 and $-(c_1 + c_2)$. The eigenspace, associated with the null eigenvalue, is tangent to Ω_4, leading to the conclusion that Ω_4 is a hyperbolic manifold.

Now, we prove that $\Omega_{4,\epsilon}$ is locally invariant under the flow expressed in equation (4.4). Therefore, the following expression holds:

$$\begin{aligned}
\psi_1' &= m_2 - \epsilon \\
\psi_2' &= 1 - \epsilon \\
\psi_3' &= -\left(c_1 + c_2\right) Y_4^{\frac{1}{p-1}} \\
\psi_4' &= -\left(c_1 + c_2\right) Y_4^{\frac{1}{p-1}} - \left[hX_3 + (1-h)X_4 \right] \epsilon.
\end{aligned} \tag{4.18}$$

Note that the above right hand functions are $C^j(\bar{B}_{r_2(0)} \times (0, \delta_1)), j > 0$ in the proximity of critical point $(m_2, 0, 1, 0)$.

In the same manner, δ_1 is assessed based on assumption that the functions in equation (4.18) are measurable in $B_{r_2}(0)$. Then, we obtain:

$$\left\| \psi_4^{\prime \Omega_4} - \psi_4^{\prime \Omega_{4,\epsilon}} \right\| \leqslant \delta_1 \epsilon, \tag{4.19}$$

where δ_1 is defined as per the Fenichel conditions, and already specified in equation (4.14).

In this section, we show that manifolds Ω_3 and Ω_4 remain invariant under the flow given in equation (4.4). Consequently, the travelling waves profiles can be obtained considering the dimensionally reduced Ω_3 and Ω_4 or the asymptotic ones $\Omega_{3,\epsilon}$ and $\Omega_{4,\epsilon}$.

4.2. Profiles of travelling waves

To determine a travelling wave profile, first, we consider the perturbed manifold $\Omega_{3,\epsilon}$ in the phase space (X_3, Y_3). A family of trajectories in this phase space can be obtained by solving the following equation:

$$\frac{\mathrm{d}Y_3}{\mathrm{d}X_3} = -\left(c_1 + c_2\right) - \left[hX_3 + (1-h)X_4\right] \epsilon Y_3^{-\frac{1}{p-1}} = H\Big(X_3, X_4, Y_3\Big), \tag{4.20}$$

$0 < X_3 < m_2,\ 0 < X_4 < 1;\ Y_3 > 0.$

To obtain the existence of a regular trajectory in the proximity of the critical point $\big(m_2, 0, 1, 0\big)$ in the phase plane $\big(X_3, Y_3\big)$, we use the condition $h(1-m_2) \leqslant 1$. For this, we utilize the continuous condition of H. We also consider certain cases in the involved parameters $c_1,\ c_2$ and ϵ. The following assessment holds for $h(1-m_2) \leqslant 1$:

$$-\Big[hX_3 + (1-h)X_4\Big] \epsilon \leqslant 0. \tag{4.21}$$

For $c_1 + c_2 < 0$ and $\epsilon > 0$, we obtain $\frac{\mathrm{d}Y_3}{\mathrm{d}X_3} > 0$ in the proximity of the critical point. If we consider $c_1 + c_2 > 0$, we find that $\frac{\mathrm{d}Y_3}{\mathrm{d}X_3} < 0$. Functions of the form of $H(X_3, Y_3)$ are continuous in their domains. Based on this topological principle, we state that there exists a regular trajectory in phase plane (X_3, Y_3) expressed as

$$-\left(c_1 + c_2\right) - \left[hX_3 + (1-h)X_4\right] \epsilon Y_3^{-\frac{1}{p-1}} = H\Big(X_3, X_4, Y_3\Big) = 0, \tag{4.22}$$

in the proximity of the critical point.

In addition, if we consider that the trajectory evolves in the proximity of $X_4 = 1$, the following expression holds

$$\big(c_1 + c_2\big) Y_3^{\frac{1}{p-1}} + hX_3 \epsilon = -(1-h)\epsilon. \tag{4.23}$$

A particular form of the travelling wave profile is obtained by considering the density and flux functions introduced in equation (4.3)

$$F_1'(\zeta) + \frac{h\epsilon}{c_1 + c_2} F_1(\zeta) = -\frac{(1-h)\epsilon}{c_1 + c_2}. \tag{4.24}$$

After solving equation (4.24), we have

$$F_1(\zeta) = \frac{1-h}{h} + E_8 e^{-\frac{h\epsilon}{c_1 + c_2}\zeta}. \tag{4.25}$$

For simplicity, we consider $E_8 = 1$, such that

$$F_1(\zeta) = \frac{1-h}{h} + e^{-\frac{h\epsilon}{c_1 + c_2}\zeta}. \tag{4.26}$$

After obtaining F_1, the same procedure is repeated for the manifold $\Omega_{4,\epsilon}$. The trajectory in phase plane $\left(X_4, Y_4\right)$ can be obtained by solving the following equation:

$$F_2'(\zeta) + \frac{(1-h)\epsilon}{c_1+c_2}(F_2(\zeta) = -\frac{hm_2\epsilon}{c_1+c_2}. \tag{4.27}$$

After solving equation (4.27), we obtain

$$F_2(\zeta) = -\frac{hm_2\epsilon}{1-h} + E_9 e^{-\frac{(1-h)\epsilon}{c_1+c_2}\zeta}. \tag{4.28}$$

For simplicity, we take $E_9 = 1$, then we have

$$F_2(\zeta) = -\frac{hm_2\epsilon}{1-h} + e^{-\frac{(1-h)\epsilon}{c_1+c_2}\zeta}. \tag{4.29}$$

4.3. Numerical assessments

The objective of this section is to provide a numerical assessment for systems presented in equations (4.1) and (4.2) and to compare such an approach with the analytical expressions obtained based on the geometric perturbation theory expressed in equations (4.26) and (4.29).

Notably, numerical assessments are not intended to build a technical discussion regarding the forms of the solutions depending on each of the parameters involved in the flame behaviour given in equations (4.1) and (4.2). Alternatively, the idea was to test the consistency of the expressions obtained based on the analytical approaches followed, which ended in solutions expressed in equations (4.26) and (4.29). This assessment allowed us to validate the following hypothesis.

In addition, it is of paramount interest to determine the forms of the solutions for different travelling wave speeds. This facilitates the determination of whether particular values of $c_{2,\min}$ (for purely monotonic profiles in the spirit of the classical KPP problem) can be obtained. To render the travelling wave speed relevant over other forms of transport mechanisms given by advection, it is sufficient to make c_1 negligible compared with c_2.

Consequently, the following parametrical values are assumed:

$$h = 0.5,\ p = 3,\ c_1 \ll c_2,\ \epsilon \leqslant 10^{-3}. \tag{4.30}$$

The numerical simulation was performed using the MATLAB function *bvp4c* (refer to [38] for a complete description of). The initial data are considered to have a step like distribution: $u(z,0) = H(-z)$, $v(z,0) = H(-z)$ where H is the Heaviside function. We consider that the initial data belongs to L^∞ and is sufficiently general with no additional, yet somehow artificial, regularity conditions. In addition, we can consider that the propagating wave moves from $z < 0$ and then enters into the porous domain, which can be assumed to appear for $z \geqslant 0$. Hence, the primary question to be answered is the form of the propagating tail once a wave motion occurs in the porous domain.

The domain of integration is given by $D = 1000$ such that the impact of the collocation method, employed by the *bvp4c*, is minimized. The considered number of nodes is 10 000, and the absolute error level is of 10^{-4}.

The results and comparisons are shown in figures 1–6 for different values of the common travelling wave speed, which should be understood as a parameter for discussing the forms of the solutions. Notably, an oscillating condition of the solutions appeared for small values of c_2. However, when the travelling wave speed increased, the travelling wave profiles exhibited a purely monotonic behaviour. The travelling speed values are provided in each figure footprint. In addition, the exponential profiles obtained in equations (4.26) and (4.29) follow the

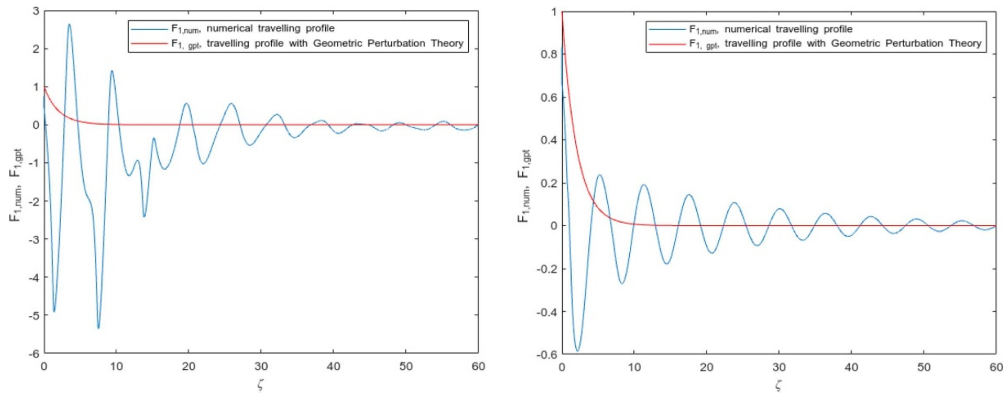

Figure 1. Representation of solutions for $c_2 = 0.01$ (left) and $c_2 = 0.1$ (right), and under the condition of $c_1 \ll c_2$. Note that for these mentioned values in the travelling wave speeds, the numerical solutions for the systems of equations (4.1) and (4.2) oscillate, whereas the solutions of form of equation (4.26), obtained based on the geometric perturbation theory, preserve monotone behaviour.

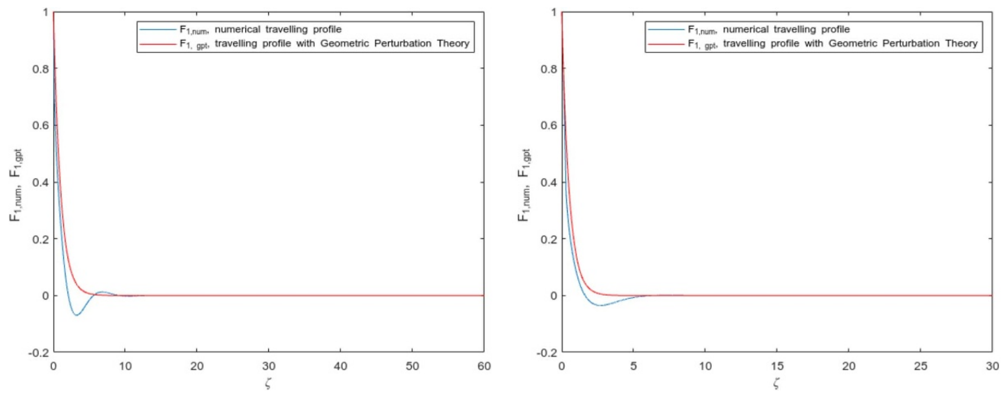

Figure 2. Representation of solutions for $c_2 = 1$ (left) and $c_2 = 1.5$ (right), and under the condition of $c_1 \ll c_2$. Note that for these values in the travelling wave speeds, the numerical solutions for systems of equations (4.1) and (4.2) oscillate, whereas the solutions of form of equation (4.26), obtained based on the geometric perturbation theory, preserve a monotone behaviour.

same decreasing behaviour as the numerical solutions. These results validate the hypothesis made when introducing the travelling wave approach and constructing solutions based on the geometric perturbation theory.

If the advection coefficient increases, and according to the system of equations (4.1) and (4.2), the monotone behaviour in F_1 is observed for $c_1 + c_2 = 1.824$ (this value was obtained for the case of $c_1 \ll c_2$ in figure 3) and in F_2 it is observed for $c_1 + c_2 = 1.758$ (as obtained for the case of $c_1 \ll c_2$ in figure 6). We can select a travelling wave speed that may counteract the effect of advection. For example, for F_1 (similar for F_2), if $c_1 > 1.824$ ($c_1 > 1.758$ for F_2), the travelling wave speed will be negative to compensate for the effect of advection and preserve the global monotone property of each solution.

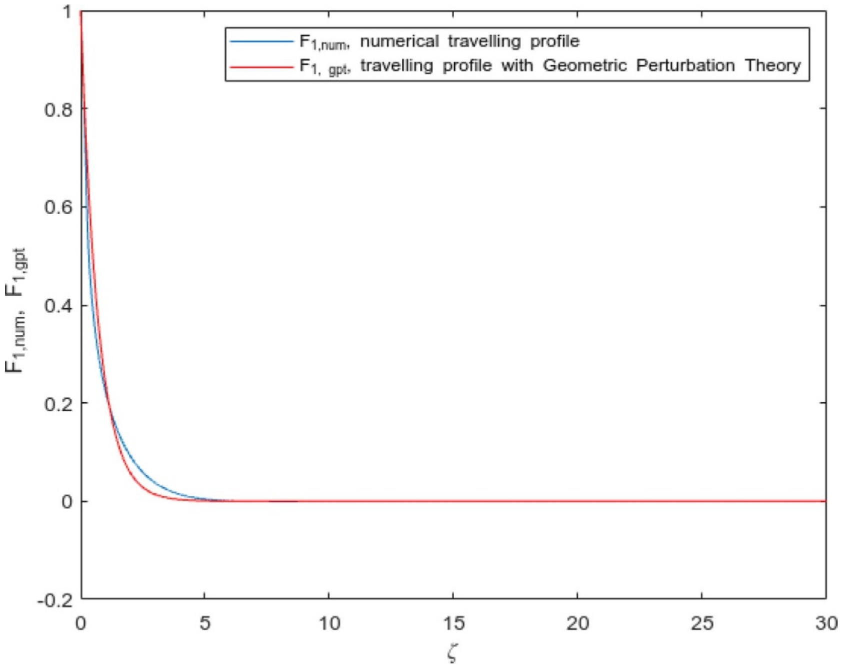

Figure 3. Representation of solutions for $c_2 = 1.824 = c_{2,\min}$. Note that for this value in the travelling wave speed, the numerical solution for the equation (4.1) is purely monotone and maintains the exponential form as obtained in equation (4.26) based on the geometric perturbation theory. The global error between both solutions was obtained as $< 9.321 \times 10^{-3}$.

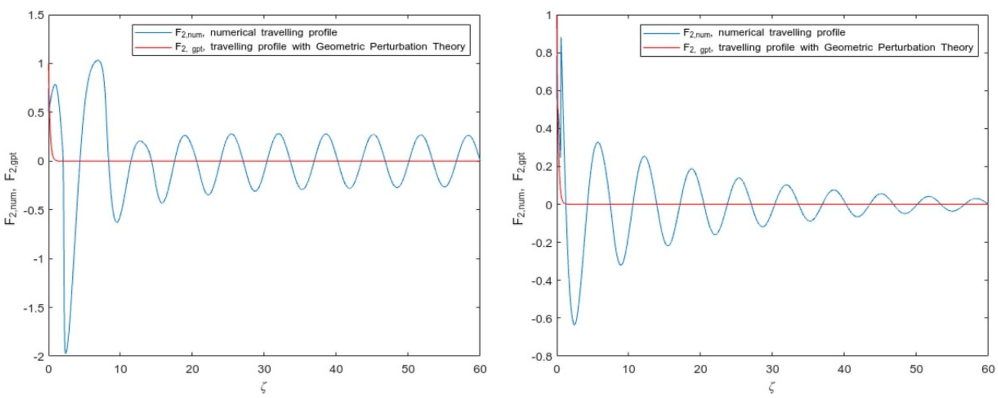

Figure 4. Representation of solutions for $c_2 = 0.01$ (left) and $c_2 = 0.1$ (right), and under the condition of $c_1 \ll c_2$. Note that for these mentioned values in the travelling wave speeds, the numerical solutions for the system of equations (4.1) and (4.2) oscillate, whereas the solutions of the form of equation (4.29), obtained based on the geometric perturbation theory, preserve a monotone behaviour.

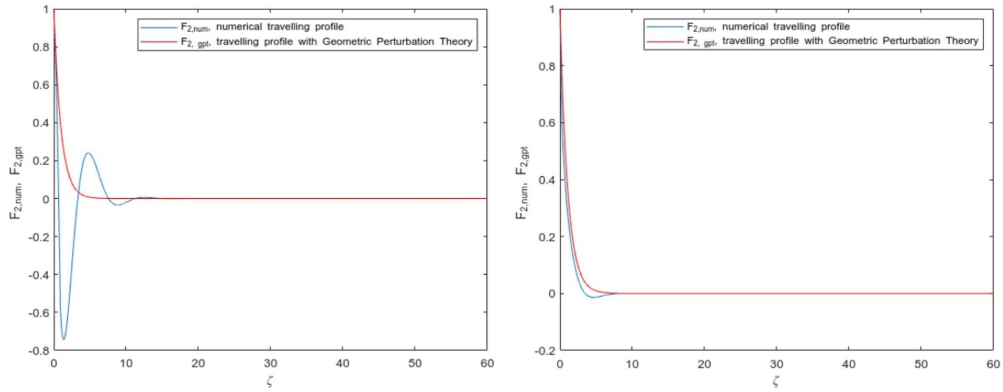

Figure 5. Representation of solutions for $c_2 = 1$ (left) and $c_2 = 1.5$ (right) and under the condition of $c_1 \ll c_2$. Note that for these values in the travelling wave speeds, the numerical solutions for the system of equations (4.1) and (4.2) oscillate, whereas the solutions of the form of equation (4.29), obtained based on the geometric perturbation theory, preserve a monotonic behaviour.

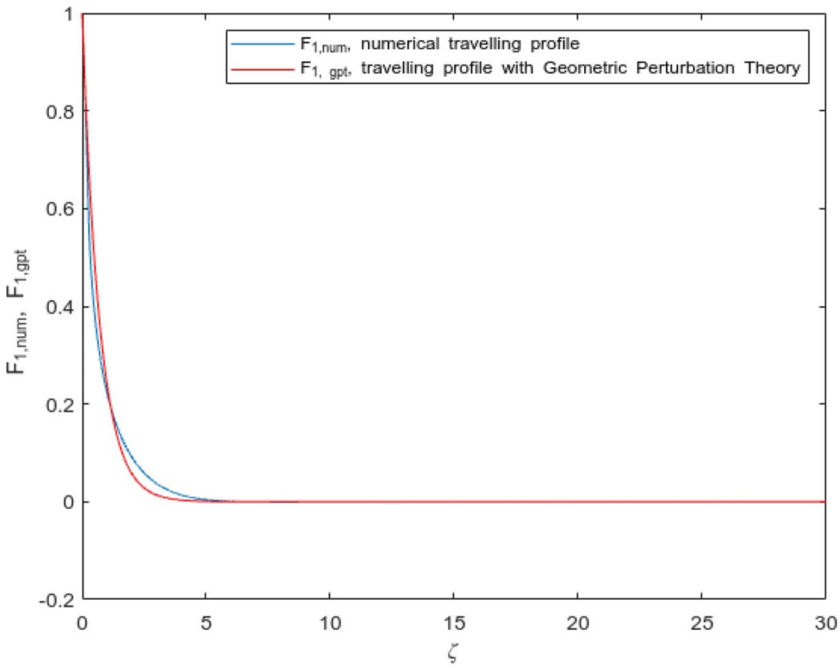

Figure 6. Representation of solutions for $c_2 = 1.758 = c_{2,\min}$. Note that for this value in the travelling wave speed, the numerical solution for the equation (4.1) is purely monotone and keeps the exponential form obtained in equation (4.29) based on the geometric perturbation theory. The global error between both solutions was obtained as $< 7.258 \times 10^{-3}$.

5. Profiles of analytical solutions and propagating supports

In this section, we explore the behaviour of the solutions to equation (1.13). We provide analytical expressions and introduce a form of propagating support for each involved solution. This propagating support provides the evolution of flame descriptors (u, v) in the (z, t) domain, leading to the formation of traces, that characterize and follow the evolution of the flame. We studied self-similar solutions and considered an arbitrarily small advection.

We start by analysing the upper solutions of system of equation (1.13) such that the following assessment holds:

$$\begin{aligned}\frac{\partial u}{\partial t} &\leqslant \frac{\partial}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^{p-2}\frac{\partial u}{\partial z}\right)+c_1\frac{\partial u}{\partial z}+w\\ &\leqslant \frac{\partial}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^{p-2}\frac{\partial u}{\partial z}\right)+c_1\frac{\partial u}{\partial z}+hu+v\\ &\leqslant \frac{\partial}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^{p-2}\frac{\partial u}{\partial z}\right)+c_1\frac{\partial u}{\partial z}+E_{10}u,\end{aligned} \tag{5.1}$$

where $E_{10}u = \sup\{hu+v\}$ and

$$\begin{aligned}\frac{\partial v}{\partial t} &\leqslant \frac{\partial}{\partial z}\left(\left(\frac{\partial v}{\partial z}\right)^{p-2}\frac{\partial v}{\partial z}\right)+c_1\frac{\partial v}{\partial z}+w\\ &\leqslant \frac{\partial}{\partial z}\left(\left(\frac{\partial v}{\partial z}\right)^{p-2}\frac{\partial v}{\partial z}\right)+c_1\frac{\partial v}{\partial z}+hu+v\\ &\leqslant \frac{\partial}{\partial z}\left(\left(\frac{\partial v}{\partial z}\right)^{p-2}\frac{\partial v}{\partial z}\right)+c_1\frac{\partial v}{\partial z}+E_{11}v\end{aligned} \tag{5.2}$$

where $E_{11}v = \sup\{hu+v\}$. The solutions to the following equations provide the upper solutions to equation (1.13):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^{p-2}\frac{\partial u}{\partial z}\right)+c_1\frac{\partial u}{\partial z}+E_{10}u, \tag{5.3}$$

and

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial z}\left(\left(\frac{\partial v}{\partial z}\right)^{p-2}\frac{\partial v}{\partial z}\right)+c_1\frac{\partial v}{\partial z}+E_{11}v. \tag{5.4}$$

The self-similar structures of the above system can be generally expressed as:

$$u(z,t) = t^{-\beta_1}g_1(\eta), \quad \eta = |z|t^{-\beta_2}, \tag{5.5}$$

and

$$v(z,t) = t^{-\beta_3}g_2(\eta), \quad \eta = |z|t^{-\beta_2}. \tag{5.6}$$

After substituting equations (5.5) and (5.6) into (5.3) and (5.4) respectively, we obtain

$$\begin{aligned}&-\beta_1 t^{-\beta_1-1}g_1-\beta_2 t^{-\beta_1-1}\eta g_1' = (p-1)t^{-\beta_1(p-1)-\beta_2}\left(g_1'\right)^{p-2}g_1''\\ &\quad+t^{-(\beta_1+\beta_2)(p-1)}\frac{(d-1)}{\eta}\left(g_1'\right)^{p-1}+c_1t^{-(\beta_1+\beta_2)}g_1'+E_{10}g_1,\end{aligned} \tag{5.7}$$

and

$$-\beta_3 t^{-\beta_3-1} g_2 - \beta_2 t^{-\beta_3-1}\eta g_2' = (p-1)t^{-\beta_3(p-1)-\beta_2}\left(g_2'\right)^{p-2} g_2'' \\ + t^{-(\beta_3+\beta_2)(p-1)}\frac{(d-1)}{\eta}\left(g_2'\right)^{p-1} + c_1 t^{-(\beta_3+\beta_2)} g_2' + E_{10} g_2. \tag{5.8}$$

Note that the solutions to equations (5.7) and (5.8) can be obtained at each given time; therefore, for simplicity, we consider $t = 1$. This yields

$$(p-1)\left(g_1'\right)^{p-2} g_1'' + (p-1)\frac{(d-1)}{\eta}\left(g_1'\right)^{p-1} + \left(\beta_2 + c_1\right)\eta g_1' + \left(\beta_1 + E_{10}\right) g_1 = 0, \tag{5.9}$$

and

$$(p-1)\left(g_2'\right)^{p-2} g_2'' + (p-1)\frac{(d-1)}{\eta}\left(g_2'\right)^{p-1} + \left(\beta_2 + c_1\right)\eta g_2' + \left(\beta_3 + E_{11}\right) g_2 = 0. \tag{5.10}$$

On solving equation (5.9) we obtain

$$\eta^{(p-1)(d-1)}(g_1')^{p-1} + \left(\beta_2 + c_1\right)\eta^{(p-1)(d-1)+1} g_1 = E_{12}, \tag{5.11}$$

where $\beta_2 + c_1 = \frac{\beta_1 + E_8}{(p-1)(d-1)+1}$. Because $g_1 \to 0$, when $\eta \to \infty$, we can set $E_{12} = 0$. Consequently, equation (5.11) becomes

$$\eta^{(p-1)(d-1)}(g_1')^{p-1} + \left(\alpha_2 + c_1\right)\eta^{(p-1)(d-1)+1} g_1 = 0. \tag{5.12}$$

On solving (5.12), we obtain

$$g_1(\eta) = \left(E_{13} - \frac{p-2}{p-1}\left(-\beta_2 - c_1\right)^{\frac{p}{p-1}}\eta^{\frac{p}{p-1}}\right)^{\frac{p-2}{p-1}}. \tag{5.13}$$

On solving equation (5.10) in a manner similar to that when solving equation (5.9), we obtain

$$g_2(\eta) = \left(E_{14} - \frac{p-2}{p-1}(-\beta_2 - c_1)^{\frac{p}{p-1}}\eta^{\frac{p}{p-1}}\right)^{\frac{p-2}{p-1}}. \tag{5.14}$$

Substituting equations (5.13) and (5.14) into equations (5.5) and (5.6), we obtain

$$u(z,t) = t^{-\beta_1}\left(E_{13} - \frac{p-2}{p-1}(-\beta_2 - c_1)^{\frac{p}{p-1}}\left(zt^{-\beta_2}\right)^{\frac{p}{p-1}}\right)^{\frac{p-2}{p-1}}, \tag{5.15}$$

and

$$v(z,t) = t^{-\beta_2}\left(E_{14} - \frac{p-2}{p-1}(-\beta_2 - c_1)^{\frac{p}{p-1}}\left(zt^{-\beta_2}\right)^{\frac{p}{p-1}}\right)^{\frac{p-2}{p-1}}. \tag{5.16}$$

The propagating support is obtained by considering

$$u(z,t) = 0 \text{ and } v(z,t) = 0. \tag{5.17}$$

Subsequently, the supports propagate under the following expressions for the (z,t) variables:

$$|z| = t^{\beta_1}\left(\frac{p-1}{p-2}\right)^{\frac{p-1}{p}}\frac{E_{13}^{\frac{p-1}{p}}}{\beta_2 + c_1}, \tag{5.18}$$

for the u-solution, and

$$|z| = t^{\beta_3} \left(\frac{p-1}{p-2}\right)^{\frac{p-1}{p}} \frac{E_{14}^{\frac{p-1}{p}}}{\beta_2 + c_1}, \tag{5.19}$$

for the v-solution.

6. Conclusions

In the analysis performed in this study, we proposed a new model to characterize the behaviour of temperature- and pressure-driven flames. For this purpose, we introduced a p-Laplacian operator with an advection term and a nonlinear reaction supported by a linear form of flame kinetics. Our analyses began by showing the uniqueness and boundedness of the weak solutions. The profiles of the solutions were studied based on the travelling waves approach and geometric perturbation theory. Consequently, the minimum travelling wave speeds for purely monotonic and exponential profiles were shown to hold. The assumptions considered to obtain the analytical solutions were validated through a numerical assessment. In addition, numerical values for the minimum travelling wave speed, $c_{2,\min}$, have been provided. For any $c_2 \geqslant c_{2,\min}$ (and for each of the profiles associated to u and v), the solutions were monotonic and followed an exponential evolution, as predicted by the application of the geometric perturbation theory. Eventually, we obtained the evolution of the flame support for the solutions u and v. As the main limitation of our study, we stress the need to conduct suitable experiments for which the obtained analytical and numerical solutions can be further useful.

Data availability statement

No data is associated with this paper. The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

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Conflict of interest

The authors declare no competing interest.

Authors' contributions

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References

[1] IPCC 2007 Climate change *Synthesis Report (IPCC Plenary XXVII) (Valencia, Spain, 12–17 November 2007)*
[2] Kamal M M and Mohamad A A 2006 Combustion in porous media *Proc. Inst. Mech. Eng.* A **220** 487–508
[3] Viskanta R 2005 Combustion and heat transfer in inert porous media *Handbook of Porous Media* 2nd edn (CRC Press) pp 607–44 (available at: www.taylorfrancis.com/chapters/edit/10.1201/9780415876384-18/combustion-heat-transfer-inert-porous-media-raymond-viskanta)
[4] Viskanta R 2011 Modeling of combustion in porous inert media *Spec. Top. Rev. Porous Media* **2** 181–204
[5] Oliveira A A M and Kaviany M 2001 Nonequilibrium in the transport of heat and reactants in combustion in porous media *Prog. Energy Combust. Sci.* **27** 523–45
[6] Scheffer M and Colombo P 2005 *Cellular Ceramics: Structure, Manufacturing, Properties and Applications* (Wiley-VCH)
[7] Grosse J, Dietrich B, Garrido G I, Habisreuther P, Zarzalis N, Martin H, Kind M and Kraushaar-Czarnetzki B 2009 Morphological characterization of ceramic sponges for applications in chemical engineering *Ind. Eng. Chem. Res.* **48** 10395–401
[8] Trimis D and Durst F 1996 Combustion in a porous medium-advances and applications *Combust. Sci. Technol.* **121** 153–68
[9] Mujeebu M A, Abdullah M Z, Bakar M Z A, Mohamad A A and Abdullah M K 2009 Applications of porous media combustion technology a review *Appl. Energy* **86** 1365–75
[10] Mujeebu M A, Abdullah M Z, Bakar M Z A, Mohamad A A, Muhad R M N and Abdullah M K 2009 Combustion in porous media and its applications a comprehensive survey *J. Environ. Manage.* **90** 2287–312
[11] Kolmogorov A N, Petrovsky I G and Piskunov N S 1937 Etude de I'equation de la diffusion avec croissance de la quantite de matiere et son application a un probleme biologique *Moskow Univ. Math. Bull.* **1** 1–25
[12] Fisher R A 1937 The wave of advance of advantageous genes *Ann. Eugen.* **7** 355–69
[13] Schelkin K I 1940 Influence of the tube walls roughness on the onset and propagation of detonation in gases *Sov. J. Exp. Theor. Phys.* **10** 823–7
[14] Brailovsky I, Goldshtein V, Shreiber I and Sivashinsky G 1997 On combustion waves driven by diffusion of pressure *Combust. Sci. Technol.* **124** 145–65
[15] Sivashinsky G I 2002 Some developments in premixed combustion modeling *Proc. Combust. Inst.* **29** 1737–61
[16] Wu Z, Zhao J, Yin J and Li H 2001 *Nonlinear Diffusion Equations* (Word Scientific)
[17] Williams F A 1985 *Combustion Theory* (Perseus Books)
[18] Zeldovich Y B, Barenblatt G I, Librovich V B and Makhviladze G M 1985 *Mathematical Theory of Combustion and Explosion* (Plenum)
[19] Mallordy J-F and Roquejoffre J-M 1995 A parabolic equation of the KPP type in higher dimensions *SIAM. J. Math. Anal.* **26** 1–20
[20] Gordon P 2006 Quenching and propagation of combustion fronts in porous media *Commun. Math. Sci.* **4** 471–9
[21] Gordon P 2007 Recent mathematical results on combustion in hydraulically resistant porous media *Math. Modelling Nat. Phenom.* **2** 56–76
[22] Ghazaryan A and Gordon P 2008 KPP type flame fronts in porous media *Nonlinearity* **21** 973

[23] Billingham J and Needham D J 1991 The development of travelling waves in quadratic and cubic autocatalysis with unequal diffusion rates: I. Permanent form travelling waves *Phil. Trans. R. Soc.* A **334** 1–24
[24] Smreker O 1879 Das Grundwasser und seine Verwendung zu Wasserversorgungen *Z. Ver. Dtsch. Ing.* **23** 347–62
[25] Smreker O 1878 Entwicklung eines Gesetzes für den Widerstand bei der Bewegung des Grundwassers *Z. Ver. Dtsch Ing.* **22** 117–28
[26] Benedikt J, Girg P, Kotrla L and Takac P 2018 Origin of the p-Laplacian and A. Missbach *Electron. J. Differ. Equ.* **2018** 1–17
[27] Kamin S and Vazquez J L 1988 Fundamental solutions and asymptotic behaviour for the p-Laplacian equation *Rev. Mat. Iberoam.* **4** 339–54
[28] Bardi M 1987 Asymptotic spherical symmetry of the free boundary in degenerate diffusion equations *Ann. Mat. Pura Appl.* **148** 117–30
[29] Bellemare L 2000 *A Multiphase Model to Predict Forest Fire Behaviour* (Universite de Provence)
[30] Morandini F, Santoni P A, Balbi J H, Ventura J and Mendes-Lopes J 2002 A two-dimensional model of fire spread across a fuel bed including wind combined with slope conditions *Int. J. Wildland Fire* **11** 53–63
[31] Chorin A J 1980 Flame advection and propagation algorithms *J. Comput. Phys.* **35** 1–11
[32] Long E and Hargrave G 2011 Experimental measurement of local burning velocity within a rotating flow *Flow Turbul. Combust.* **86** 455–76
[33] de Pablo A and Vazquez J L 1991 Travelling waves and finite propagation in a reaction-diffusion equation *J. Differ. Equ.* **93** 19–61
[34] Bebernes J and Schmitt K 1979 On the existence of upper and lower solutions for parabolic partial differential equations *Proc. Am. Math. Soc.* **73** 211–8
[35] Fenichel N 1971 Persistence and smoothness of invariant manifolds for flows *Indiana Univ. Math. J.* **21** 193–226
[36] Akveld M E and Hulshof J 1998 Travelling wave solutions of a fourth-order semilinear diffusion equation *Appl. Math. Lett.* **11** 115–20
[37] Jones C K 1995 *Geometric Singular Perturbation Theory in Dynamical Systems* (Springer)
[38] Enright H and Muir P H 1993 A Runge-Kutta type boundary value ODE solver with defect control *Technical Report* 267/93 (University of Toronto, Department of Computer Sciences)