Insight into the Eyring-Powell fluid flow model using degenerate operator: geometric perturbation

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Abstract

This work provides a formulation of a fluid flow under a nonlinear diffusion based on a viscosity of Eyring-Powell type along with a degenerate semi-parabolic operator. The introduction of such a degenerate operator is significant as it allows us to explore a further general model not previously considered in the literature. Our aims are hence to provide analytical insights and numerical assessments to the mentioned flow model: Firstly, some results are provided in connection with the regularity and uniqueness of weak solutions. The problem is converted into the travelling wave domain where solutions are obtained within an asymptotic expansion supported by the Geometric Perturbation Theory. Finally, a numerical process is considered as the basis to ensure the validity of the analytical assessments presented. Such numerical process is performed for low Reynolds numbers given in classical porous media. As a main finding to highlight: we show that there exist exponential profiles of solutions for the velocity component. This result is not trivial for the non-linear viscosity terms considered.

Keywords: Eyring-Powell fluid, Porous Medium Equation, Travelling waves, Geometric Perturbation Theory.

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1 Introduction

The Fluid movements under non-Newtonian descriptions are relevant in biomedical applications, food processes, chemical industries, construction and engineering connecting with the physical principles of fluids. The classical Navier-Stokes equations require additional considerations when discussing non-Newtonian flows to formulate the constitutive equations describing their rheological properties. The mathematical description of transport processes in porous media was initially introduced in 1856 by Darcy [14] for the analysis of water flows in sand beds. The reduced diameter in the interstitial structure of a porous medium makes the fluid to move in low velocities leading to small Reynolds numbers. The increase of such Reynolds requires to increase the pressure gradients in the medium to account for increased velocity profiles. This fact motivates the existence of turbulence in the fluid leading to make the Darcy‘s law vague. To avoid this issue, firstly in 1901 Forcheheimer [13] and then in 1956 Jaeger [11], proposed empirical relations between the velocity and the hydraulic gradient in a turbulent flow. Based on the studies by Forcheheimer [13] and Jaeger [11], the Darcy‘s laws was renamed as Darcy-Forchheimer‘s principle by Mustak in [7] and Ward in [12]. Note that the mentioned formulations were considered for Newtonian fluids. Nonetheless, the Darcy-Forchheimer‘s law has been employed for non-Newtonian fluids as well (the reader is referred to the studies by Pal and Mondal [17], Ganesh *et al* [18], Haq *et al* [19] and Hayat *et al* [20] and references therein for a wider scope). In the present paper, we introduce a novelty to model in porous media as an alternative to the Darcy-Forchheimer law. Our idea is to introduce a non-linear diffusion typically leading to the well known equation of Porous Medium. Such an equation exhibits the property of finite propagation of inherent interest for modelling flows in porous domains where the velocity behaves in a non-regular fashion (refer to Pablo [1] and Pablo and Vázquez [4]).

Of the several types of non-Newtonian fluids existing in the literature, we focus our analysis in the Eyring-Powell fluid flow. This model allows to describe the shear stress and the associated viscosity based on the kinetic theory of liquids. The underlying theory is robust enough to describe a precise kinematic condition for viscosity. In addition, the fluid is assumed to flow in a porous medium. To describe such condition, it is possible to employ a hybrid Eyring-Powell and Darcy-Forchheimer flow. This approach has been explored in Abbas and Megahed [10] to model a fluid in a stratified stretching sheet embedded in a porous medium. In this mentioned study, numerical solutions, supported by a Chebyshev spectral method, are obtained to the set of non-linear equations describing the flow under study. Alternatively, our intention is to make use of mathematical properties, like the regularity and uniqueness of weak solutions along with analytical and numerical solutions. To put the Eyring-Powell fluid flow into a proper context, we provide some interesting studies with applications of Eyring-Powell fluis flow: The Eyring-Powel fluid model has been considered to model flow profiles in magnetohydrodynamics (MHD). Akbar *et al.* [15] studied solutions for a two dimensional MHD fluid. In addition, Hina [21] analyzed a MHD Eyring-Powell fluid with heat transfer. Later, Bhatti et.al [16] considered a permeable stretching surface for an MHD flow and made an analysis involving heat transfer. In Abegunrin *et al.* [2], the authors investigated a two-dimensional Eyring-Powell fluid to describe the flow past paraboloid shaped bodies. They obtained solutions based on a fourth order Runge-Kutta scheme along with shooting

techniques. In Koriko *et al.* [3], the authors provided analyses to understand the effects of nonlinear thermal radiation and stratification for a three-dimensional flow of Eyring-Powell in a 36-nm alumina-water nanofluid. Furthermore, in Animasaun *et al.* [5], the authors provided the analysis of a general flow over neither a perfect horizontal nor vertical and neither an inclined surface nor a cone/wedge. The resulting equations were solved with a numerical approach supported by a Runge–Kutta procedure with shooting. Finally, the reader is referred to the studies by Ara *et al.* [22], Hayat *et al.* [23], Hayat *et al.* [24], Jalil *et al.* [25], Murray [26], Smoller [27], Champneys *et al.* [28] and Riaz *et al.* [29] for further analyses concerning Eyring-Powell fluids.

The equation under discussion in this work has not been previously considered in the literature. Hence, the provided model needs to be studied first from a mathematical approach to assert its correctness for future analyses and applications. Such mathematical approach starts with the characterization of solutions, and in particular with the boundedness and uniqueness of general solutions (given in the frame of a weak mathematical formulation). It is important to remark that in coming sections we will consider that solutions do exist. Although we do not provide a whole mathematical assessment based, for example, on a successive approximation method, the reader is referred to Sections 4 and 5 where we introduce travelling waves profiles via analytical and numerical procedures. These results can serve as an non-formal proof of existence of solutions. It is particularly relevant to state about the exponential kind of solutions expressed in (4.14), and this is actually an interesting result given the non-linear operator discussed in this work.

As it will be described, the present paper provides travelling wave solutions. In this regards, and in the search of analytical profiles to non-Newtonian fluids, the studies by Murray [26], Smoller [27] and Champneys *et al.* [28] provide important results. In our case, the convergence and stability results in the travelling wave profiles are provided by a Geometric Perturbation approach and are validated by a numerical procedure.

The paper structure is: Firstly, a discussion about the fluid model proposed is introduced. Afterward, regularity of solutions is explored. Profiles of solutions are obtained based on the theory of travelling waves and the Geometric Perturbation theory. A numerical process aims at validating the followed analytical approach. In addition, an optimal travelling wave speed is discussed. Such optimal propagation speed provides the minimal global quadratic error between the asymptotic solution and the numerical profile. Finally, the existence of an exponential travelling wave profile is shown to hold.

2 Mathematical model description

Let us consider the flow of one dimensional, unidirectional, incompressible, unsteady Eyring-Powell fluid in the presence of a magnetic field. The velocity field is of the form $\mathbf{V} = \big((u(y,t),0,0\big)$. Hina [21] derive the main equations of two dimensional and two directional unsteady flow of Eyring-Powell fluid. Considering a velocity

vector as $\mathbf{V} = (u(y,t), 0, 0)$ then the equations use in Hina [21] become

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= -\frac{1}{\rho_f}\frac{\partial P}{dx} + \left(\nu + \frac{1}{\beta d_1 \rho_f}\right)\frac{\partial^2 u_1}{\partial y^2} - \frac{1}{2\beta d_1^3 \rho_f}\left(\frac{\partial u_1}{\partial y}\right)^2 \frac{\partial^2 u_1}{\partial y^2} - \frac{\sigma B_0^2 u_1}{\rho_f} \\ 0 &= -\frac{\partial p}{\partial y} \end{aligned} \tag{2.1}$$

where μ refers to the dynamic viscosity, ρ_f is the density, $\nu = \frac{\mu}{\rho_f}$ is the kinematic viscosity, P refers to the pressure which depends only on x, $u_1(y,t)$ is the first component of velocity, B_0 means the imposed magnetic field and β and d_1 are two constants characteristic of an Eyring-Powell fluid. Let us introduce the following non-dimensional quantities based on Eldabe *et al.* [31]:

$$\begin{aligned} x^* &= \frac{x}{L},\ y^* = \frac{y}{L},\ t^* = \frac{U_0}{L}t,\ P^* = \frac{P}{\rho_f U_0^2},\ u_1^* = \frac{u_1}{U_0}, \\ A_1 &= \frac{1}{R_e},\ A_2 = \frac{1}{\beta d_1 \mu},\ \epsilon = \frac{U_0}{2\rho_f \beta d_1^3 L^3}, \end{aligned} \tag{2.2}$$

where U_0 is a velocity magnitude, $R_e = \frac{U_0 L}{\nu}$ is the Reynolds number and $M = B_0 L\sqrt{\frac{\sigma}{\rho_f \nu}}$ is the Hartmann number that compares the magnetic drag forces with the viscous forces. After substitution of the set of above expressions into Eq. (2.1) (ignoring * for simplicity)

$$\frac{\partial u_1}{\partial t} = -\frac{dP}{dx} + A_1(1+A_2)\frac{\partial^2 u_1}{\partial y^2} - \epsilon\left(\frac{\partial u_1}{\partial y}\right)^2 \frac{\partial^2 u_1}{\partial y^2} - \frac{M^2}{R_e}u_1. \tag{2.3}$$

Note that A_1 and A_2 are fluid constants, that are generally introduced for the sake of simplicity in the notation. The diffusion is extended to account for a non-linear term of the form $\frac{\partial^2 u_1^m}{\partial y^2}$, where $m > 1$ which describes the slow diffusion effect on Eyring-Powell fluid flow (again see Pablo [1] and Pablo and Vázquez [4] for a wide discussion)

$$\frac{\partial u_1}{\partial t} = -\frac{dP}{dx} + A_1(1+A_2)\frac{\partial^2 u_1^m}{\partial y^2} - \epsilon\left(\frac{\partial u_1}{\partial y}\right)^2 \frac{\partial^2 u_1}{\partial y^2} - \frac{M^2}{R_e}u_1. \tag{2.4}$$

Since u_1 is a function of y and t, we differentiate Eq. (2.3) with regards to x to get

$$-\frac{d^2 P}{dx^2} = 0.$$

After integration we get

$$-\frac{dP}{dx} = A_3,$$

where A_3 is any constant. After using this value for $-\frac{dP}{dx}$, the following holds:

$$\frac{\partial u_1}{\partial t} = A_3 + A_1(1+A_2)\frac{\partial^2 u_1^m}{\partial y^2} - \epsilon\left(\frac{\partial u_1}{\partial y}\right)^2 \frac{\partial^2 u_1}{\partial y^2} - \frac{M^2}{R_e}u_1. \tag{2.5}$$

The related initial condition aims at generalizing a non-regular initial distribution of a fluid. To this end:

$$u_1(y,0) = u_{10}(y) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{2.6}$$

3 Regularity and uniqueness of solutions

The regularity of solutions is analyzed based on a weak formulation of (2.5) given the degenerate operator considered. In addition, the use of the weak solution concept is relevant to relax any mathematical strong condition that may not be realistic in a non-regular porous medium. For this, assume a test function $\Phi \in C^{\infty}(\mathbb{R} \times I)$ where $I = [\tau, T]$, $0 < \tau < t < T$ and $\Phi(y, \tau) = \Phi(y, T) = 0$. The following weak formulation holds

$$\begin{aligned}\int_{\mathbb{R}} u_1(t)\,\Phi(t)\,dy &= \int_{\mathbb{R}} u_1(\tau)\,\Phi(\tau)\,dy + \int_{\tau}^{t}\int_{\mathbb{R}} u_1 \frac{\partial \Phi}{\partial s} dyds + A_3 \int_{\tau}^{t}\int_{\mathbb{R}} \Phi dyds \\ &+ A_1(1+A_2)\int_{\tau}^{t}\int_{\mathbb{R}} u_1^m \frac{\partial^2 \Phi}{\partial y^2} dyds + \frac{\epsilon}{3}\int_{\tau}^{t}\int_{\mathbb{R}} \left(\frac{\partial u_1}{\partial y}\right)^3 \frac{\partial \Phi}{\partial y} dyds \\ &- \frac{M^2}{R_e}\int_{\tau}^{t}\int_{\mathbb{R}} u_1 \Phi dyds. \end{aligned} \tag{3.1}$$

Consider a finite $r_0 \in \mathbb{R}^+$ and define a ball B_r centered at the origin of coordinates and such that $r \gg r_0$. In this domain B_r, the following equation is defined

$$u_1 \frac{\partial \Phi}{\partial s} + A_3 \Phi + A_1(1+A_2)\, u_1^m \frac{\partial^2 \Phi}{\partial y^2} + \frac{\epsilon}{3}\left(\frac{\partial u_1}{\partial y}\right)^3 \frac{\partial \Phi}{\partial y} - \frac{M^2}{R_e} u_1 \Phi = 0, \tag{3.2}$$

supplemented with the following pseudo-boundary like condition (for the test function) and initial distribution

$$0 < \frac{\partial \Phi}{\partial y} = \Phi \ll 1 \;\text{ for }\; |y| \to \infty, \;\; u_1(y,0) = u_{10}(y) \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R}).$$

The intention is to study the boundedness and regularity of solutions for the problem defined in Eq. (3.2), and we introduce for these purposes the *Theorems 3.1, 3.2.*

Theorem 3.1. *Given $u_{10}(y) \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, then any positive solution, $u_1(y,t) \geq 0$, is bounded for $(y,t) \in B_r \times [\tau, T]$ with $\tau > 0$ and $r \gg 1$.*

Proof. Consider $\gamma \in \mathbb{R}^+$, the following cut-off function is defined

$$\psi_{\gamma} \in C_0^{\infty}(y,t), \qquad 0 \leq \psi_{\gamma} \leq 1$$

$$\psi_{\gamma} = 1 \text{ in } B_{r-\gamma}, \quad \psi_{\gamma} = 0 \;\text{ in }\; \mathbb{R} - B_{r-\gamma},$$

that complies with

$$\left|\frac{\partial \psi_{\gamma}}{\partial \gamma}\right| = \frac{A_b}{\gamma},$$

being $A_b > 0$ a suitable constant.

Multiplying the Eq. (3.2) by ψ_γ and integrating in $B_r \times [\tau, t]$, with $0 < \tau < s \leq t \leq T$

$$\int_\tau^t \int_{B_r} u_1 \frac{\partial \Phi}{\partial s} \psi_\gamma dy ds + A_3 \int_\tau^t \int_{B_r} \Phi \psi_\gamma dy ds = -A_1 (1 + A_2) \int_\tau^t \int_{B_r} u_1^m \frac{\partial^2 \Phi}{\partial y^2} \psi_\gamma dy ds$$

$$- \frac{\epsilon}{3} \int_\tau^t \int_{B_r} \left(\frac{\partial u_1}{\partial y} \right)^3 \frac{\partial \Phi}{\partial y} \psi_\gamma dy ds + \frac{M^2}{R_e} \int_\tau^t \int_{B_r} u_1 \Phi \psi_\gamma dy ds. \tag{3.3}$$

Note that for some $r \gg r_0 > 1$ with $y \in (0, r)$, the following holds (see Pablo [1] and Pablo and Vázquez [4])

$$\int_\tau^t u_1 ds \leq \int_\tau^t u_1^m ds \leq B_1 r^{\frac{2m}{m-1}}, \tag{3.4}$$

where $B_1 > 0$ is a constant.

In what follows, consider that the spatial variable y is close to the boundary ∂B_r, and under this condition we may consider both variables y and r as equivalent. In addition, we use the variable r for introducing appropriate bounds, while the variable y keeps the notion of variable in the inner region of B_r, but close to ∂B_r.

From (3.4), we obtain

$$\int_\tau^t \frac{\partial}{\partial y} (u_1^m) ds \leq \frac{2m}{(m-1)} B_1 r^{\frac{m+1}{m-1}}.$$

and

$$\int_\tau^t \frac{\partial u_1}{\partial y} ds \leq \frac{2m}{(m-1)} B_1 r^{\frac{m+1}{m-1}}.$$

Now, based on the Hölder inequality

$$\int_\tau^t \frac{\partial u_1}{\partial y} ds \leq \left(T^{\frac{2}{3}} - \tau^{\frac{2}{3}} \right) \left[\int_\tau^t \left(\frac{\partial u_1}{\partial y} \right)^3 ds \right]^{\frac{1}{3}},$$

where $0 < \tau < s \leq t \leq T$. Then:

$$- \left[\int_\tau^t \left(\frac{\partial u_1}{\partial y} \right)^3 ds \right]^{\frac{1}{3}} \leq - \frac{1}{T^{\frac{2}{3}} - \tau^{\frac{2}{3}}} \int_\tau^t \frac{\partial u_1}{\partial y} ds,$$

which implies that

$$- \int_\tau^t \left(\frac{\partial u_1}{\partial y} \right)^3 ds \leq - \frac{1}{T^{\frac{2}{3}} - \tau^{\frac{2}{3}}} \left[\int_\tau^t \frac{\partial u_1}{\partial y} ds \right]^3 \leq \left[\int_\tau^t \frac{\partial u_1}{\partial y} ds \right]^3 \leq \frac{8m^3}{(m-1)^3} B_1^3 r^{\frac{3m+3}{m-1}}. \tag{3.5}$$

Now, the following integration by parts holds

$$-A_1(1+A_2)\int_\tau^t\int_{B_r} u_1^m\frac{\partial^2\Phi}{\partial y^2}\psi_\gamma dyds = -A_1(1+A_2)\int_\tau^t\Big[\left(u_1^m\frac{\partial\Phi}{\partial y}\psi_\gamma\right)_{\partial B_r} - \int_{B_r} u_1^m\frac{\partial\Phi}{\partial y}\frac{\partial\psi_\gamma}{\partial y}dy - \int_{B_r}\frac{\partial\Phi}{\partial y}\frac{\partial}{\partial y}(u_1^m)\psi_\gamma dy\Big]ds.$$

Considering a fluid relaxation condition of the form $u_1^m \to 0$ over ∂B_r for $r \gg 1$, we can state that

$$\begin{aligned}-A_1(1+A_2)\int_\tau^t\int_{B_r} u_1^m\frac{\partial^2\Phi}{\partial y^2}\psi_\gamma dyds &\le A_1(1+A_2)\int_{B_r} B_1 r^{\frac{2m}{m-1}}\frac{\partial\Phi}{\partial y}\frac{\partial\psi_\gamma}{\partial y}dy \\ &+ \frac{2mA_1(1+A_2)}{m-1}\int_{B_r} B_1 r^{\frac{m+1}{m-1}}\frac{\partial\Phi}{\partial y}\psi_\gamma dy,\end{aligned} \tag{3.6}$$

and

$$-\frac{\epsilon}{3}\int_\tau^t\int_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^3\frac{\partial\Phi}{\partial y}\psi_\gamma dyds \le \frac{8\epsilon m^3}{3(m-1)^3}\int_{B_r} B_1^3 r^{\frac{3m+3}{m-1}}\frac{\partial\Phi}{\partial y}\psi_\gamma dy. \tag{3.7}$$

Using (3.6) and (3.7), the right hand side of (3.3) becomes

$$-A_1(1+A_2)\int_\tau^t\int_{B_r} u_1^m\frac{\partial^2\Phi}{\partial y^2}\psi_\gamma dyds - \frac{\epsilon}{3}\int_\tau^t\int_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^3\frac{\partial\Phi}{\partial y}\psi_\gamma dyds + \frac{M^2}{R_e}\int_\tau^t\int_{B_r} u_1\Phi\psi_\gamma dyds$$

$$\begin{aligned}&\le A_1(1+A_2)\int_{B_r} B_1\, r^{\frac{2m}{m-1}}\frac{\partial\Phi}{\partial y}\frac{\partial\psi_\gamma}{\partial y}dy + \frac{2mA_1(1+A_2)}{m-1}\int_{B_r} B_1 r^{\frac{m+1}{m-1}}\frac{\partial\Phi}{\partial y}\psi_\gamma dy \\ &+\frac{8\epsilon m^3}{3(m-1)^3}\int_{B_r} B_1\, r^{\frac{3m+3}{m-1}}\frac{\partial\Phi}{\partial y}\psi_\gamma dy + \frac{M^2}{R_e}\int_{B_r} B_1\, r^{\frac{2m}{m-1}}\Phi\psi_\gamma dy \\ &\le A_1(1+A_2)\int_{B_r} B_1\, r^{\frac{2m}{m-1}}\frac{\partial\Phi}{\partial y}\frac{A_b}{\gamma}dy + \frac{2mA_1(1+A_2)}{m-1}\int_{B_r} B_1 r^{\frac{m+1}{m-1}}\frac{\partial\Phi}{\partial y}\psi_\gamma dy \\ &+\frac{8\epsilon m^3}{3(m-1)^3}\int_{B_r} B_1\, r^{\frac{3m+3}{m-1}}\frac{\partial\Phi}{\partial y}\psi_\gamma dy + \frac{M^2}{R_e}\int_{B_r} B_1\, r^{\frac{2m}{m-1}}\Phi dy \\ &\le A_1(1+A_2)A_b\int_{B_r} B_1\, r^{\frac{m+1}{m-1}}\frac{\partial\Phi}{\partial y}dy + \frac{2mA_1(1+A_2)}{m-1}\int_{B_r} B_1 r^{\frac{m+1}{m-1}}\frac{\partial\Phi}{\partial y}\psi_\gamma dy \\ &+\frac{8\epsilon m^3}{3(m-1)^3}\int_{B_r} B_1\, r^{\frac{3m+3}{m-1}}\frac{\partial\Phi}{\partial y}\psi_\gamma dy + \frac{M^2}{R_e}\int_{B_r} B_1\, r^{\frac{2m}{m-1}}\Phi dy.\end{aligned} \tag{3.8}$$

Next, consider a test function Φ complying with the previously stated condition: $\Phi(y,\tau) = \Phi(y,T) = 0$

$$\Phi(y,s) = e^{-ks}\left(1+y^2\right)^{-\beta}(s-\tau)(T-s). \tag{3.9}$$

Note that the exponent β is chosen in a way that the integrals in (3.8) are convergent, therefore

$$-A_1\left(1+A_2\right)\int_\tau^t\int_{B_r}u_1^m\frac{\partial^2\Phi}{\partial y^2}\psi_\gamma dyds-\frac{\epsilon}{3}\int_\tau^t\int_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^3\frac{\partial\Phi}{\partial y}\psi_\gamma dyds+\frac{M^2}{R_e}\int_\tau^t\int_{B_r}u_1\Phi\psi_\gamma dyds$$

$$\begin{aligned}
&\leq 2\beta A_1\left(1+A_2\right)A_bB_1\int_{B_r}e^{-ks}\big(s-\tau\big)\big(T-s\big)y^{\frac{2}{m-1}-2\beta}dy\\
&+\frac{2mA_1\left(1+A_2\right)}{m-1}\int_{B_r}B_1\big(s-\tau\big)\big(T-s\big)y^{\frac{m+1}{m-1}-2\beta}dy\\
&+\frac{8\epsilon m^3}{3\left(m-1\right)^3}B_1\int_{B_r}e^{-ks}\big(s-\tau\big)\big(T-s\big)y^{\frac{3m+3}{m-1}-2\beta}dy\\
&+\frac{M^2}{R_e}B_1\int_{B_r}e^{-ks}\big(s-\tau\big)\big(T-s\big)y^{\frac{2m}{m-1}-2\beta}dy.
\end{aligned}$$

Considering that $\beta>\frac{m}{m-1}$ and $y\to\infty$,

$$-A_1\left(1+A_2\right)\int_\tau^t\int_{B_r}u_1^m\frac{\partial^2\Phi}{\partial y^2}\psi_\gamma dyds-\frac{\epsilon}{3}\int_\tau^t\int_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^3\frac{\partial\Phi}{\partial y}\psi_\gamma dyds+\frac{M^2}{R_e}\int_\tau^t\int_{B_r}u_1\Phi\psi_\gamma dyds\leq D_r, \tag{3.10}$$

where D_r is a constant that involves a small mass at $y\to\infty$ in accordance with the decreasing test function tail as expressed in (3.9).

Putting (3.10) in (3.3), we obtain

$$\int_\tau^t\int_{B_r}u_1\frac{\partial\Phi}{\partial s}\psi_\gamma dyds+A_3\int_\tau^t\int_{B_r}\Phi\psi_\gamma dyds\leq D_r.$$

This last expression permits to state that the lastly pointed integrals are finite in the domain of integration and including the case of $y\to\infty$ in large domains B_r. Note that as per initial assumption, solutions are positive in $\tau<s<t<T$. Based on this last expression, it is possible to conclude on the boundedness of solutions in $B_r\times[\tau,T]$.

□

The next intention is to show the regularity of solutions based on a boundedness condition for $\frac{\partial u_1}{\partial y}$.

Theorem 3.2. *Assume the existence of a bounded solution $u_1\left(y,t\right)\geq 0$ complying with Theorem 3.1, then $\frac{\partial u_1}{\partial y}$ is bounded in $(y,t)\in Q_T=B_r\times[0,T]$, $T\geq 1$ and possibly $r\gg 1$.*

Proof. Multiplying Eq. (2.5) by u_1 and using integration by parts over the spatial domain B_r, we have

$$\begin{aligned}\frac{d}{dt}\int\limits_{B_r}|u_1|^2\,dy &= A_3\int\limits_{B_r}u_1dy+A_1\left(1+A_2\right)\int\limits_{B_r}u_1^m\frac{\partial^2u_1}{\partial y^2}dy\\ &\quad+\frac{\epsilon}{3}\int\limits_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^4dy-\frac{M^2}{R_e}\int\limits_{B_r}|u_1|^2\,dy,\end{aligned}$$

which implies that

$$\frac{\epsilon}{3}\int\limits_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^4dy+A_1(1+A_2)\int\limits_{B_r}u_1^m\frac{\partial^2u_1}{\partial y^2}dy=\frac{d}{dt}\int\limits_{B_r}|u_1|^2\,dy$$

$$-A_3\int\limits_{B_r}u_1dy+\frac{M^2}{R_e}\int\limits_{B_r}|u_1|^2\,dy.$$

After integration in both sides in $[0,t]$ with $t\leq T$,

$$\int\limits_0^t\int\limits_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^4dyds+A_1(1+A_2)\int\limits_0^t\int\limits_{B_r}u_1^m\frac{\partial^2u_1}{\partial y^2}dyds=\int\limits_{B_r}|u_1(y,t)|^2\,dy$$

$$-\int\limits_{B_r}|u_{10}\left(y\right)|^2\,dy-A_3\int\limits_0^t\int\limits_{B_r}u_1dyds+\frac{M^2}{R_e}\int\limits_0^t\int\limits_{B_r}|u_1|^2\,dyds. \tag{3.11}$$

Based on the Theorem 3.1, we state that the right hand side integrals of Eq. (3.11) are bounded, therefore we can choose a suitable constant F_1 such that

$$\int\limits_0^t\int\limits_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^4dyds+A_1(1+A_2)\int\limits_0^t\int\limits_{B_r}u_1^m\frac{\partial^2u_1}{\partial y^2}dyds\leq F_1,$$

From Theorem 3.1, we state that the positive solution u_1 is bounded, therefore we select $F_2=\inf\limits_{(y,t)\in Q_T}(u_1^m)$ such that

$$\int\limits_0^t\int\limits_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^4dyds+A_1F_2(1+A_2)\int\limits_0^t\int\limits_{B_r}\frac{\partial^2u_1}{\partial y^2}dyds\leq F_1,$$

As A_1 and A_2 are positive, we state that

$$\int\limits_0^t\int\limits_{B_r}\left(\frac{\partial u_1}{\partial y}\right)^4dyds\leq F_1,$$

Based on the Hölder inequality, the following assessment holds for any $T \geq 1$:

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} \frac{\partial u_1}{\partial y} dyds &\leq \int_{B_r} \left[\int_0^t \left(\frac{\partial u_1}{\partial y} \right)^4 ds \right]^{\frac{1}{4}} \left[\int_0^t \left(1 \right)^{\frac{4}{3}} ds \right]^{\frac{3}{4}} dy \\
&\leq T^{\frac{3}{4}} \int_{B_r} \left[\int_0^t \left(\frac{\partial u_1}{\partial y} \right)^4 ds \right]^{\frac{1}{4}} dy \\
&\leq T \int_0^t \int_{B_r} \left(\frac{\partial u_1}{\partial y} \right)^4 dyds \leq TF_1,
\end{aligned}$$

leading to state that $\frac{\partial u_1}{\partial y}$ is bounded in $(y,t) \in B_r \times [0,T]$, for any $T \geq 1$ and possibly for $r \gg 1$.

□

Our next target is to show the uniqueness of solutions.

Theorem 3.3. *Consider that $u_{11}(y,t)$ and $u_{12}(y,t)$ as two solutions of Eq. (2.5) departing from the same initial data i.e. $u_{11}(y,0) = u_{12}(y,0) = u_{10}(y) > 0$. Then: $u_{11}(y,t) = u_{12}(y,t)$ in $Q_T = B_r \times (0,T)$, for any r and possibly $r \gg 1$.*

Proof. To prove the uniqueness of solutions, consider a test function $\psi(y,t) \in C^\infty(Q_T)$ and assume that there exist two solutions such that by initial hypothesis $u_{11}(y,t) > u_{12}(y,t)$. The following weak formulation for both solutions holds

$$\begin{aligned}
\int_{B_r} u_{11}(t)\,\psi_1(t)\,dy &= \int_{B_r} u_{11}(0)\,\psi_1(0)\,dy + \int_0^t \int_{B_r} u_{11} \frac{\partial \psi_1}{\partial s} dyds \\
&+ A_1(1+A_2) \int_0^t \int_{B_r} u_{11}^m \frac{\partial^2 \psi_1}{\partial y^2} dyds + A_3 \int_0^t \int_{B_r} \psi_1 dydt \\
&+ \frac{\epsilon}{3} \int_{B_r} \left(\frac{\partial u_{11}}{\partial y} \right)^4 \frac{\partial \psi_1}{\partial y} dyds - \frac{M^2}{R_e} \int_0^t \int_{B_r} u_{11} \psi_1 dydt, \quad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
\int_{B_r} u_{12}(t)\,\psi_1(t)\,dy &= \int_{B_r} u_{12}(0)\,\psi_1(0)\,dy + \int_0^t \int_{B_r} u_{12} \frac{\partial \psi_1}{\partial s} dyds \\
&+ A_1(1+A_2) \int_0^t \int_{B_r} u_{12}^m \frac{\partial^2 \psi_1}{\partial y^2} dyds + A_3 \int_0^t \int_{B_r} \psi_1 dydt \\
&+ \frac{\epsilon}{3} \int_{B_r} \left(\frac{\partial u_{12}}{\partial y} \right)^4 \frac{\partial \psi_1}{\partial y} dyds - \frac{M^2}{R_e} \int_0^t \int_{B_r} u_{12} \psi_1 dydt. \quad (3.13)
\end{aligned}$$

Making Eq. (3.12) minus (3.13)

$$\begin{aligned}\int_{B_r}(u_{11}-u_{12})(t)\,\psi_1(t)\,dy &= \int_0^t\int_{B_r}(u_{11}-u_{12})(t)\,\frac{\partial\psi_1}{\partial s}dyds\\ &+A_1(1+A_2)\int_0^t\int_{B_r}(u_{11}^m-u_{12}^m)\,\frac{\partial^2\psi_1}{\partial y^2}dyds\\ &+\frac{\epsilon}{3}\int_0^t\int_{B_r}\left(\left(\frac{\partial u_{11}}{\partial y}\right)^3-\left(\frac{\partial u_{12}}{\partial y}\right)^3\right)\frac{\partial\psi_1}{\partial y}dyds\\ &-\frac{M^2}{R_e}\int_0^t\int_{B_r}(u_{11}-u_{12})\,\psi_1 dyds.\end{aligned} \tag{3.14}$$

For $u_{11}(y,t) > u_{12}(y,t)$, the following inequality holds

$$(u_{11}^m-u_{12}^m)\le mu_{11}^{m-1}(u_{11}-u_{12})\le mC_1^{m-1}|u_{11}-u_{12}|, \tag{3.15}$$

where $C_1=\max_{(y,t)\in Q_T}(u_{11})$. Now consider

$$\begin{aligned}\left(\left(\frac{\partial u_{11}}{\partial y}\right)^3-\left(\frac{\partial u_{12}}{\partial y}\right)^3\right) &= \left(\frac{\partial u_{11}}{\partial y}-\frac{\partial u_{12}}{\partial y}\right)\left(\left(\frac{\partial u_{11}}{\partial y}\right)^2+\frac{\partial u_{11}}{\partial y}\frac{\partial u_{12}}{\partial y}+\left(\frac{\partial u_{12}}{\partial y}\right)^2\right)\\ &\le C_2\left(\frac{\partial u_{11}}{\partial y}-\frac{\partial u_{12}}{\partial y}\right),\end{aligned} \tag{3.16}$$

where we have used the Theorem 3.2 to ensure the boundedness of the involved terms, in particular $C_2=\max_{(y,t)\in Q_T}\{\frac{\partial u_{11}}{\partial y},\frac{\partial u_{12}}{\partial y}\}$.

Setting (3.15) and (3.16) in Eq. (3.14), we have

$$\begin{aligned}\int_{B_r}(u_{11}-u_{12})(t)\,\psi_1(t)\,dy &\le \int_0^t\int_{B_r}(u_{11}-u_{12})(t)\,\frac{\partial\psi_1}{\partial s}dyds\\ &+\left(mC_1^{m-1}A_1(1+A_2)-\frac{\epsilon C_2}{3}\right)\int_0^t\int_{B_r}|u_{11}-u_{12}|\,\frac{\partial^2\psi_1}{\partial y^2}dyds\\ &-\frac{M^2}{R_e}\int_0^t\int_{B_r}(u_{11}-u_{12})\,\psi_1 dyds.\end{aligned} \tag{3.17}$$

Now, assume the following test function complying with $\psi_1(y,0)=\psi_1(y,T)=0$ for $0<s<t<T$

$$\psi_1(y,s)=e^{-ps}\left(1+|y|^2\right)^{-\beta}s(T-s), \tag{3.18}$$

for any $p > 0$ and where β shall be selected such that

$$\int_{B_r} \psi_1(y,s)\,dy < \infty. \tag{3.19}$$

In the sake of simplicity, we can choose $\int_{B_r} \psi_1(y,s)\,dy = 1$. For $y \to \infty$, the test function mass shall be small according to the expression in (3.18):

$$\int_{||y|-B_r|\to\infty} \psi_1(y,s)\,dy \to 0.$$

In the asymptotic $|y| \to \infty$, we have $|y|^{-2\beta}|y| \to 0$, then it suffices

$$\beta > \frac{1}{2}. \tag{3.20}$$

Now, let us define

$$\int_{B_r} \psi_1(y,s)\,dy = s\big(T-s\big)e^{-ps}\int_{B_r}\frac{1}{\big(1+|y|^2\big)^{\beta}}dy = s\big(T-s\big)e^{-ps}\Psi(y), \tag{3.21}$$

where

$$\Psi(y) = \int_{B_r}\frac{1}{\big(1+|y|^2\big)^{\beta}}dy. \tag{3.22}$$

The integral (3.21) is finite under the condition for β given in (3.20). Also, for $|y| \to \infty$, we obtain

$$\begin{aligned}\frac{\partial^2\psi_1}{\partial y^2} &= \frac{4\beta(\beta+1)y^2 s\big(T-s\big)e^{-ps}}{\big(1+|y|^2\big)^{\beta+2}} - \frac{2\beta s\big(T-s\big)e^{-ps}}{\big(1+|y|^2\big)^{\beta+1}} \\ &\le \frac{4\beta(\beta+1)e^{-ps}}{\big(1+|y|^2\big)^{\beta}} \le 4\beta(\beta+1)\psi_1.\end{aligned} \tag{3.23}$$

After integrating (3.23), we have

$$\int_{B_r}\frac{\partial^2\psi_1(y,s)}{\partial y^2}dy \le \int_{B_r} C_3(\beta)\psi_1(y,s)\,dy = C_3(\beta)\,s\big(T-s\big)e^{-ps}\Psi(y), \tag{3.24}$$

where $C_3(\alpha) = 4\beta(\beta+1)$. Now differentiate the test function ψ_1 with respect to s

$$\begin{aligned}\frac{\partial\psi_1}{\partial s} &= -\frac{ps\big(T-s\big)e^{-ps}}{\big(1+y^2\big)^{\beta}} + \frac{\big(T-s-s\big)e^{-ps}}{\big(1+y^2\big)^{\beta}} \\ &\le \frac{ps\big(T-s\big)e^{-ps}}{\big(1+y^2\big)^{\beta}} + \frac{s\big(T-s\big)e^{-ps}}{\big(1+y^2\big)^{\beta}} \\ &\le \big(p+1\big)\psi_1.\end{aligned}$$

The idea is to evaluate each of the integrals involved in (3.14), so that

$$\begin{aligned}
&\int_0^t\int_{B_r}(u_{11}-u_{12})(t)\frac{\partial\psi_1}{\partial s}dyds \leq \int_0^t\int_{B_r}\big(p+1\big)(u_{11}-u_{12})(t)\psi_1 dyds\\
\leq\ & \big(p+1\big)\sup|u_{11}-u_{12}|\,\Psi(y)\int_0^t s\big(T-s\big)e^{-ps}ds\\
=\ & \big(p+1\big)\sup|u_{11}-u_{12}|\,\Psi(y)\left(\frac{Tp+2-e^{-pt}\big(t^2p^2+tp^2T+Tp-2tp+2\big)}{p^3}\right).
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&\left(mC_1^{m-1}A_1(1+A_2)-\frac{\epsilon C_2}{3}\right)\int_0^t\int_{B_r}|u_{11}-u_{12}|\frac{\partial^2\psi_1}{\partial y^2}dyds\\
\leq\ & \left(mC_1^{m-1}A_1(1+A_2)-\frac{\epsilon C_2}{3}\right)\sup|u_{11}-u_{12}|\,C_3(\beta)\int_0^t\int_{B_r}\psi_1(y,s)\,dyds\\
=\ & \left(mC_1^{m-1}A_1(1+A_2)-\frac{\epsilon C_2}{3}\right)\sup|u_{11}-u_{12}|\,C_3(\beta)\,\Psi(y)\int_0^t s\big(T-s\big)e^{-ps}ds\\
=\ & \left(mC_1^{m-1}A_1(1+A_2)-\frac{\epsilon C_2}{3}\right)\sup|u_{11}-u_{12}|\,C_3(\beta)\,\Psi(y)\times\\
&\left(\frac{Tp+2-e^{-pt}\big(t^2p^2+tp^2T+Tp-2tp+2\big)}{p^3}\right).
\end{aligned} \tag{3.26}$$

and similarly

$$\begin{aligned}
&-\frac{M^2}{R_e}\int_0^t\int_{B_r}(u_{11}-u_{12})\psi_1 dyds\\
\leq\ & \frac{M^2}{R_e}\sup|u_{11}-u_{12}|\,\Psi(y)\left(\frac{Tp+2-e^{-pt}\big(t^2p^2+tp^2T+Tp-2tp+2\big)}{p^3}\right).
\end{aligned} \tag{3.27}$$

Combining (3.25) and (3.27) with (3.14), the following holds

$$\begin{aligned}
&\int_{B_r} (u_{11}-u_{12})(t)\,\psi_1(t)\,dy \\
&\leq \sup|u_{11}-u_{12}|\,\Psi(y)\left(\frac{Tp+2-e^{-pt}\left(t^2p^2+tp^2T+Tp-2tp+2\right)}{p^3}\right) \\
&+\left(mC_1^{m-1}A_1(1+A_2)-\frac{\epsilon C_2}{3}\right)\sup|u_{11}-u_{12}|\,C_3(\beta)\,\Psi(y)\times \\
&\left(\frac{Tp+2-e^{-pt}\left(t^2p^2+tp^2T+Tp-2tp+2\right)}{p^3}\right) \\
&+\frac{M^2}{R_e}\sup|u_{11}-u_{12}|\,\Psi(y)\left(\frac{Tp+2-e^{-pt}\left(t^2p^2+tp^2T+Tp-2tp+2\right)}{p^3}\right).
\end{aligned}$$

Since $|\Psi(y)|<\infty$ for $t>0$ and in the assumption of a sufficiently close condition of the form $\sup|u_{11}-u_{12}|\to 0^+$, we obtain

$$0<\int_{B_r}(u_{11}-u_{12})(t)\,\psi_1(t)\,dy\leq\epsilon^+,$$

where ϵ^+ is a small and positive bounding constant complying with $\epsilon^+\to 0^+$ whenever $\sup|u_{11}-u_{12}|\to 0^+$. Then we state that $u_{11}(t)\to u_{12}(t)$ (asymptotic uniqueness) in $Q_T=B_r\times(0,T)$ for any r and possibly $r\gg 1$. In addition, if we make the exact condition $\sup|u_{11}-u_{12}|=0$, then:

$$\int_{B_r}(u_{11}-u_{12})(t)\,\psi_1(t)\,dy=0,$$

leading to state the exact uniqueness of solutions in $Q_T=B_r\times(0,T)$.

□

4 Travelling waves solutions

A travelling wave profile is given by by $u_1(y,t)=h(\eta)$, where $\eta=y-ct\in\mathbb{R}$, c refers to the travelling wave speed and $h:\mathbb{R}\to(0,\infty)$ is the travelling wave profile that needs to be determined.

Making use of the travelling waves formulation, the Eq. (2.5) is then transformed in terms of the travelling wave profile as

$$-c\,h'=A_3+A_1(1+A_2)(h^m)''-\epsilon(h')^2h''-\frac{M^2}{R_e}h, \tag{4.1}$$

Let us consider the new variables

$$X=h(\eta),\qquad Y=(h^m(\eta))', \tag{4.2}$$

the following system holds

$$\begin{aligned} X' &= \frac{1}{m}X^{1-m}Y \\ Y' &= \frac{m^3}{m^3A_1\left(1+A_2\right)-\epsilon X^{3-3m}Y^2}\left[-\frac{c}{m}X^{1-m}Y+\frac{\epsilon\left(1-m\right)}{m^3}X^{2-3m}Y^3-A_3+\frac{M^2}{R_e}X\right]. \end{aligned} \tag{4.3}$$

To find the critical points, we set $X'=0$ and $Y'=0$, so that

$$X=\frac{A_3R_e}{M^2}.$$

As a consequence, the only critical point is $\left(\frac{A_3R_e}{M^2},0\right)$. Our intention is to use the Geometric Perturbation Theory to characterize the mentioned critical point and to explore the orbits in its proximity.

4.1 Geometric Perturbation Theory

In this section we make use of the singular geometric perturbation theory to show the asymptotic behaviour of the following manifold defined as

$$\begin{aligned} N_0 &= \{X,Y\ /\ X'=\frac{1}{m}X^{1-m}Y; \\ Y' &= \frac{m^3}{m^3A_1\left(1+A_2\right)-\epsilon X^{3-3m}Y^2}\left[-\frac{c}{m}X^{1-m}Y+\frac{\epsilon\left(1-m\right)}{m^3}X^{2-3m}Y^3-A_3+\frac{M^2}{R_e}X\right]\}, \end{aligned}$$

with critical point $\left(\frac{A_3R_e}{M^2},0\right)$. The perturbed manifold N_α close to N_0 in the critical point $\left(\frac{A_3R_e}{M^2},0\right)$ is defined as:

$$N_\alpha=\left\{X,Y\ /\ X'=\alpha Y;\ Y'=K_1\alpha\left(X-\frac{A_3R_e}{M^2}\right)\right\}, \tag{4.4}$$

where α denotes a perturbation parameter close to equilibrium $\left(\frac{A_3R_e}{M^2},0\right)$ and K_1 is a suitable constant found after root factorization. Firstly, we let $X_1=X-\frac{A_3R_e}{M^2}$. The idea is to apply the Fenichel invariant manifold Theorem (see Fenichel [6]) as formulated by Jones [8]. For this purpose, we have to show that N_0 is a normally hyperbolic manifold, i.e. the eigenvalues of N_0 in the linearized frame close to the critical point, and transversal to the tangent space, have non-zero real part. This is shown based on the following equivalent flow associated to N_α

$$\begin{pmatrix} X_1' \\ Y' \end{pmatrix}=\begin{pmatrix} 0 & \alpha \\ K_1\alpha & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ Y \end{pmatrix}$$

The associated eigenvalues are both real $\left(\pm\sqrt{K_1\alpha}\right)$, which show that N_0 is a hyperbolic manifold. Now we shall show that the manifold N_α is locally invariant under the flow (4.3), so that the manifold N_0 can be shown as an asymptotic approach to N_α. For this basis we consider the functions

$$\begin{aligned} \psi_1 &= \alpha Y \\ \psi_2 &= K_1\alpha X_1 \end{aligned}$$

which are $C^i\left(R\times[0,\delta]\right)$ $(i\geq 1)$ in the proximity of the critical point $\left(\frac{A_3R_e}{M^2},0\right)$. In this case, δ is determined based on the following flows that are considered to be measurable almost everywhere in $\mathbb{R}$

$$\left\|\psi_1^{N_0}-\psi_1^{N_\alpha}\right\|\leq K_1\alpha\left\|X_1\right\|\leq\delta\alpha,$$

Since the solutions have been shown to be bounded, it is possible to define $\delta = K_1\left\|X_1\right\|$ and to ensure that is finite. As a consequence, the distance between the manifolds holds the normal hyperbolic condition for $\delta\in(0,\infty)$ and for α sufficiently small, close the critical point $\left(\frac{A_3R_e}{M^2},0\right)$.

4.2 Travelling Waves Profiles

Based on the previously shown normal hyperbolic condition in the manifold N_0 under the flow (4.3), it is possible to obtain convergent asymptotic travelling wave profiles.

For this purpose, let us consider firstly (4.3) such that the following family of trajectories in the phase plane (X,Y) holds

$$\begin{aligned}\frac{dY}{dX} &= \frac{m^3}{\left(m^3A_1\left(1+A_2\right)-\epsilon X^{3-3m}Y^2\right)X^{1-m}Y}\left[-\frac{c}{m}X^{1-m}Y+\frac{\epsilon\left(1-m\right)}{m^3}X^{2-3m}Y-A_3+\frac{M^2}{R_e}X\right]\\ &= H\left(X,Y\right).\end{aligned}\tag{4.5}$$

We search for a critical trajectory of the form

$$-\frac{c}{m}X^{1-m}Y+\frac{\epsilon\left(1-m\right)}{m^3}X^{2-3m}Y-A_3+\frac{M^2}{R_e}X=0,$$

which implies that

$$\epsilon\left(1-m\right)\left(X'\right)^3-cXX'-A_3X+\frac{M^2}{R_e}X^2=0.\tag{4.6}$$

Assume that a solution to the Eq.(4.6) is spanned in terms of a perturbation parameter ϵ (see Hina [21] to account for a wider discussion about perturbation approaches). In the case under study, the perturbation parameter ϵ can be described based on physical principles. Indeed, considering the low velocity values in a porous medium flow ($U_0\ll 1$), and given a large $L\gg 1$, we find in (2.2), $\epsilon\ll 1$ for classical fluid densities. As a consequence, the perturbation parameter is considered as the non-dimensional ϵ in (2.2). In addition, consider the following comparison

$$\frac{1}{Re}(1+A)\gg\frac{\epsilon}{3}\quad\rightarrow\quad\left(\frac{U_0}{L}\right)^2\ll 6\nu C_1^2(\rho_fC_1\beta+1).$$

The spanned solution is expressed as

$$X=X_0+\epsilon X_1+\epsilon^2X_2+\ldots\qquad 0<\epsilon\ll 1.\tag{4.7}$$

Substitute Eq.(4.7) into Eq. (4.6) to obtain

$$\begin{aligned}\epsilon\left(1-m\right)\left(X_0'+\epsilon X_1'+\epsilon^2X_2'+...\right)^3 &= c\left(X_0+\epsilon X_1+\epsilon^2X_2+...\right)\left(X_0'+\epsilon X_1'+\epsilon^2X_2'+...\right)\\ &+A_3\left(X_0+\epsilon X_1+\epsilon^2X_2+...\right)\\ &-\frac{M^2}{R_e}\left(X_0+\epsilon X_1+\epsilon^2X_2+...\right)^2. \end{aligned} \tag{4.8}$$

Comparing ϵ^0 and ϵ^1, we get

$$X_0'+\frac{A_3}{c}-\frac{M^2}{cR_e}X_0=0. \tag{4.9}$$

and

$$X_1'+\left(\frac{X_0'}{X_0}+\frac{A_3}{cX_0}-\frac{2M^2}{cR_e}\right)X_1=\frac{\left(1-m\right)\left(X_0'\right)^3}{c\left(X_0\right)}. \tag{4.10}$$

Solving (4.10), we obtain

$$X_0=\frac{A_3R_e}{M^2}+e^{\frac{M^2}{cR_e}\eta},$$

Note that the solution holds by symmetry invariance ($\eta\to-\eta$) of travelling wave profiles

$$X_0=\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}\eta}. \tag{4.11}$$

Putting (4.11) into (4.10), we have

$$X_1'+\left(-\frac{\frac{M^2}{cR_e}e^{-\frac{M^2}{cR_e}\eta}}{\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}\eta}}+\frac{A_3}{c\left(\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}\eta}\right)}-\frac{2M^2}{cR_e}\right)X_1=-\frac{\left(1-m\right)M^6e^{-\frac{M^2}{cR_e}\eta}}{c^4R_e^3\left(\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}\eta}\right)}, \tag{4.12}$$

After solving Eq. (4.12), we obtain

$$X_1=\frac{\left(1-m\right)cR_ee^{-\frac{3M^2}{cR_e}\eta}\left(\frac{A_3R_e}{4M^2}+\frac{1}{5}e^{-\frac{M^2}{cR_e}\eta}\right)}{M^2\left(\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}\eta}\right)}. \tag{4.13}$$

Using Eq. (4.13) and (4.11) into Eq. (4.7), we get

$$X=\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}\eta}+\frac{\epsilon\left(1-m\right)cR_ee^{-\frac{3M^2}{cR_e}\eta}\left(\frac{A_3R_e}{4M^2}+\frac{1}{5}e^{-\frac{M^2}{cR_e}\eta}\right)}{M^2\left(\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}\eta}\right)}+O\left(\epsilon^2\right)$$

$$u_1\left(y,t\right)=\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}(y-ct)}+\frac{\epsilon\left(1-m\right)cR_ee^{-\frac{3M^2}{cR_e}(y-ct)}\left(\frac{A_3R_e}{4M^2}+\frac{1}{5}e^{-\frac{M^2}{cR_e}(y-ct)}\right)}{M^2\left(\frac{A_3R_e}{M^2}+e^{-\frac{M^2}{cR_e}(y-ct)}\right)}+O\left(\epsilon^2\right) \tag{4.14}$$

As it can be seen, the existence of an exponential profile holds. This is not a trivial result for the non-linear diffusion case under study and provides evidences on the existence of an exponential profile as solution to (2.1). In addition, if $m>1$ in

(4.14), and given the small ϵ parameter, we state that the velocity profile is lower compared with a purely exponential profile (for this, it suffices to check the third element on the right hand side).

Now, the intention is to show that the defined supporting manifold N_α preserves the exponential behaviour in the proximity of the critical point. For this purpose, consider the expression (4.4), so that

$$\frac{dY}{dX} = \frac{K_1\left(X - \frac{A_3 R_e}{M^2}\right)}{Y}. \tag{4.15}$$

Solving (4.15) by using separation of variables, we obtain

$$Y = -K_1\left(X - \frac{A_3 R_e}{M^2}\right),$$

After using (4.4), the above equation becomes

$$X' = -K_1\alpha\left(X - \frac{A_3 R_e}{M^2}\right). \tag{4.16}$$

Solving (4.16)

$$X = \frac{A_3 R_e}{M^2} + e^{-K_1\alpha\eta}.$$

From (4.2), we have

$$h(\eta) = \frac{A_3 R_e}{M^2} + e^{-K_1\alpha\eta}, \qquad u_1(y,t) = \frac{A_3 R_e}{M^2} + e^{-K_1\alpha(y-ct)}.$$

This last expression permits to show the conservation of the exponential profile close the critical point defined by the asymptotic manifold N_α.

5 Numerical assessments

The analytical approach presented in the previous sections is validated making use of a numerical routine. To this end, sufficiently accurate solutions to the problem given in Eq. (4.3) are provided making use of the function *bvp4c* in Matlab. The obtained numerical solutions are compared with the analytical profiles obtained and compiled in (4.14).

This numerical approach has not the intention to establish a whole discussion for a set of different values for the parameter ϵ. On the contrary, the intention is to explore the accuracy of the geometric perturbation method introduced above that ended in the analytical profile (4.14). Certainly, dedicated values in the involved parameters have been assumed to make the validation exercise tractable. Note that the observed conclusions are somehow general, as we have computed solutions for other combinations in the involved constants with similar results compared to the ones presented.

The numerical exercise is performed according with the following key bullets:

- As discussed, the numerical routine is based on the Matlab function *bvp4c*. The *bvp4c* solver follows a Runge-Kutta implicit approach with interpolant

extensions as discussed by Enright and Muir [9]. The *bvp4c* is supported by a collocation method that requires to specify the pseudo-boundary conditions at infinity. In our case, the solution profiles are studied for $y > 0$ departing from a randomly positive u_{10}.

- To avoid the effects of the collocation process over the solutions, the integration domain is assumed to be sufficiently large. To this end, several computational pre-tests have been executed to determine the optimal domain extension. It has been concluded that a domain as $(-1000, 1000)$ provides a good compromise between moderate computational efforts and negligible influence of the collocation method effects. In addition, we have specified an initial condition satisfying (2.6). For this, we have considered the following initial data:

$$u_1(y,0) = 0 \,\text{in}\, (-\infty, -1000),\ u_1(y,0) = 1 \,\text{in}\, [-1000, 0),\ u_1(y,0) = 0 \,\text{in}\, [0, \infty). \tag{5.1}$$

- The global absolute accumulated error considered is of 10^{-5}.

Unless otherwise specified, the fluid constants in our equations and solutions are all considered to have a unit value without loss of generality. Flow in a porous medium system typically occurs at low Reynolds numbers (Buchori *et al.* [30]). Hence, during the computational exercises, the Reynolds number is considered as $Re = 1,\ 10,\ 50$.

The criteria for selecting an optimal travelling wave velocity is based on the global quadratic distance between the numerical solution and the asymptotic approximated profile. The global error shall be below 10^{-3} for $\eta > \eta_b$ where η_b is a sufficiently large value (in our case, fixed at $\eta_b = 5$). According to Figure 2, for $Re = 1$ the minimizing travelling wave speed is given for $c = 0.3701$. The same exercise has been repeated for $Re = 10$, leading to a travelling wave speed of $c = 0.4022$. Finally, proceeding similarly for $Re = 50$, the travelling wave speed that minimizes the global quadratic error is $c = 0.8845$. These last results permit to confirm that the optimal minimizing travelling wave speed is sensitive to the flow conditions, i.e. to the Reynolds number.

As it can be inferred from Figures 1 and 2, the analytical asymptotic solution obtained in (4.14) and the numerical solution to (4.3) fit an exponential behaviour. This is not a trivial conclusion given the non-linear diffusion considered and represents and interesting finding to highlight in this work.

6 Conclusions

The different sections in this paper have discussed properties of solutions to a fluid flow formulated with a viscosity of Eyring-Powell type and a non-linear diffusion of porous medium type. Firstly, regularity and uniqueness of solutions have been shown to hold. Afterward, the problem has been studied based on the theory of travelling wave solutions. The combination of travelling waves and the Geometric Perturbation Theory has permitted to obtain analytical solutions based on an asymptotic expansion. Finally, The analytical profile obtained has been validated based on a comparison (in the sense of quadratic global error) with a numerical

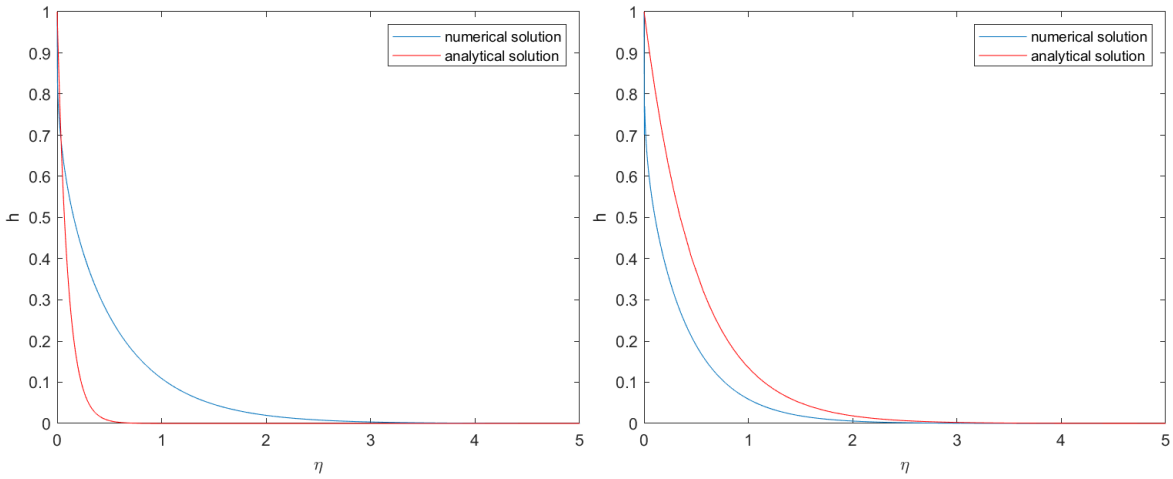


Figure 1: Travelling wave profiles for $Re = 1$, $c = 0.1$ (left) and $c = 0.5$ (right). Note that for increasing values in the travelling wave speed c, both solutions tend to get closer. Observe that the numerical solution is passed from above on the left to below on the right. This suggests that the optimal travelling wave profile is between $c = 0.1$ and $c = 0.5$. Note that the solutions are vertically translated by $-\frac{A_3 R_e}{M^2}$ to the origin.

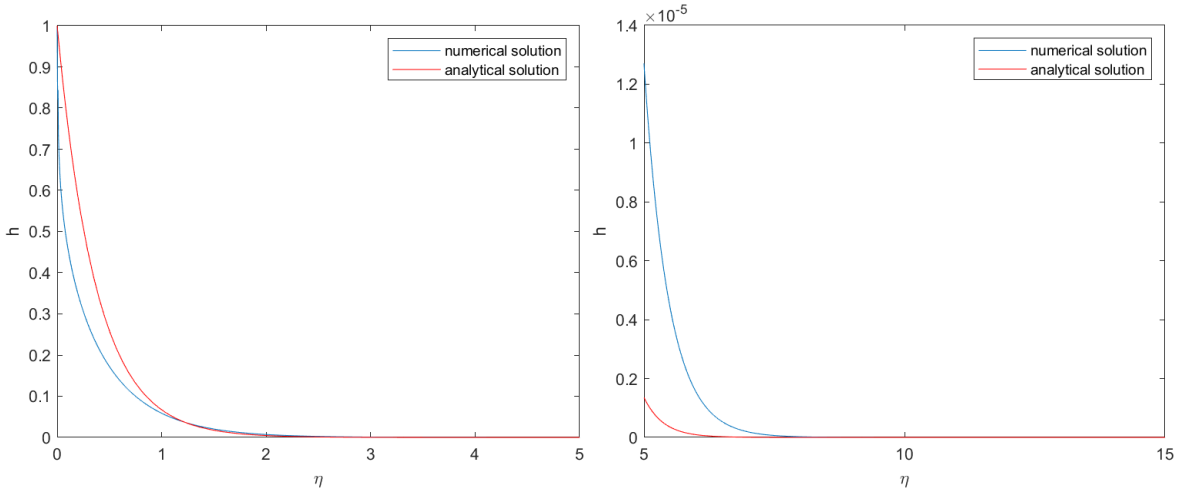


Figure 2: Solutions for $Re = 1$ and $c = 0.3701$. The global accumulative quadratic norm between the numerical and analytical solutions are below 10^{-3} for $\eta > 5$. This condition is get based on the global distance in the right figure. Note that the solutions are vertically translated by $-\frac{A_3 R_e}{M^2}$ to the origin.

solution. This idea has permitted to determine a travelling wave velocity for which the global quadratic error is minimized under different flow conditions expressed by the Reynolds number. Based on all exposed, the overall objective in connection with analytical and numerical assessments to the proposed flow can be considered as achieved. As a future prospect, there is a need for exploring dedicated applications. The general solutions and analyses discussed in this work may serve to support a first modeling exploration of physical processes concerning Eyring-Powell fluid flows.

7 Conflict of Interests

The authors declare that there is not any conflict of interests.

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