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Semigroup theory and analysis of solutions for a higher order non-Lipschitz problem with advection in $\mathbb{R}^N$

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In studying heterogeneous diffusion phenomena beyond the classical order 2 Laplacian, higher order operators emerge as powerful tools. When such operators are applied, the resultant profiles often showcase an oscillatory nature that may be noticeable near the zero stationary solution. These oscillations can pose challenges, especially when they negate the compactness of the support for initial distributions that are both smooth and possess compact support. The presented analysis in this work seeks to delve into the nature of solutions within open finite domains, with an ambition to scale up the findings to the realm of $\mathbb{R}^N$ through the use of an extension operator. The foundation for establishing the existence and uniqueness of these solutions leans on the semigroup representation theory. Asymptotic solutions have been carved out via the Hamilton–Jacobi transport equation. To cement the theoretical foundation, numerical simulations have been incorporated shedding light on the accuracy of our initial assumptions within the asymptotic framework.

KEYWORDS
Hamilton–Jacobi, higher order diffusion, non-Lipschitz, semigroup representation

MSC CLASSIFICATION
35K92, 35K91, 35K55

1 | INTRODUCTION

The realm of mathematical models in connection with reaction–diffusion phenomena has consistently acted as a breeding ground for innovative techniques in the PDE theory, seamlessly meshing with interdisciplinary applications. Physically speaking, diffusion is often articulated through the prism of the Random Walk concept, which is deeply rooted in particle interactions [1]. This framework proves instrumental in capturing the intricacies of diffusion in spatially diverse domains.

Diverging slightly, Landau and Ginzburg introduced an alternative approach by leveraging the concept of free energy [2, 3]. The essence of this approach lies in utilizing a global interaction-based generalized function rather than being tethered to specifics regarding the gradient of concentration, a stipulation generally imposed by the classical Fick law. Delving into [2], the free energy construct was adeptly employed to depict spatially distributed solutions within intricate mediums. Pivoting on the assumption that free energy hinges on a concentration gradient quadratic function, specifically $\frac{1}{2}k(\nabla u)^2$, and harnessing the idea of chemical potential, an order 4 parabolic operator materialized, predominantly of the $u_t = -u_{xxxx}$ form.

Steering the conversation back to foundational concepts, seminal works by Kolmogorov et al. [4] and Fisher [5] laid the groundwork for the mathematical analysis of reaction and diffusion problems. Rooted in the linear Fick law, they unveiled the concept of traveling wave solutions, which illuminated the dynamism underlying diffusion-centric phenomena. The sensitivity of these solutions to wave propagation speed led to fascinating discoveries, such as oscillatory behaviors. Subsequent applications and evolution of the Fisher-KPP model brought forth more intricate parabolic operators, as seen in [6–11].

Highlighting the Landau–Ginzburg approach once more, the induction of an order 4 parabolic operator through the free energy concept is noteworthy. To fortify this concept, higher order operators have been brought into the mix as extensions to the traditional order 2 diffusion, aiming to aptly model oscillatory solution profiles. Notable extensions of the Fisher–Kolmogorov equation are discussed in [12–15].

Diverse analyses have enriched our understanding of heterogeneous diffusion. For instance, [16] delves into a biological interaction enriched by an advection term, while [17] probes into haptotaxis cancer dynamics augmented by degenerate diffusivity, leveraging spectral stability methods.

The equation under discussion in this analysis is formed of a higher order operator, a nonlinear advection, and a non-Lipschitz reaction:

$$\begin{gathered} u_t = -\Delta^2 u + c \cdot \nabla u^p + u^q, \\ u_0(x) \in W_0^{m,p}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N), \quad N > 1, \end{gathered} \tag{1}$$

where $m = 4$, typically $p = 2$ and $0 < q < 1$. The advection term nonlinear order is aligned with the Lebesgue space L^p to sidestep potential issues with solutions that could become globally unbounded.

Our proposed equation is versatile, apt for modeling the spread of species within a specific domain. The higher order operator captures heterogeneous diffusion, leading to oscillatory solution patterns near the zero state. When close to this state, the non-Lipschitz reaction introduces specific irregularities, which can simulate scenarios like sudden population surges. The nonlinear advection component offers insights into primary movement patterns, which might be towards food sources or away from invasive threats. Let us delve deeper into potential interpretations and relevant references:

1. *Biharmonic operator* $\Delta^2 u$:
   - *Physical meaning*: Represents a more general diffusion compared with the order 2 Laplacian arising from the classical Fick law.
   - *Applications*:
     - *Fluid mechanics*: Appears in the study of viscous flows, especially in boundary layer theory when considering the velocity profile near walls [18].
     - *Chemistry*: After a detail review in the literature, it rarely appears. However, in the context of molecular dynamics, the biharmonic can sometimes represent higher order potentials or forces.
     - *Biology*: Membrane and cell wall deformations, especially in the mechanics of red blood cells and similar structures.
2. *Gradient term* ∇u^p:
   - *Physical meaning*: Nonlinear convection or flux.
   - *Applications*:
     - *Fluid mechanics*: Nonlinear advection terms in the Navier–Stokes equations, especially in turbulent flows where velocity magnitudes drive transport [19].
     - *Chemistry*: Nonlinear transport phenomena, where the flux of a substance is driven by a power of its concentration.
     - *Biology*: Chemotaxis, where cells move in response to concentration gradients of chemicals. This movement can be nonlinear, with the speed of movement depending on the chemical concentration itself [20].
3. *Power term* u^q:
   - *Physical meaning*: Represents sources or sinks.
   - *Applications*:
     - *Fluid mechanics*: Rarely appears explicitly, but can signify sources/sinks of vorticity or heat.
     - *Chemistry*: Reaction kinetics, especially in autocatalytic reactions where the rate of reaction is dependent on the concentration of the product or reactant raised to a power.
     - *Biology*: Growth or decay processes, such as bacterial growth where the rate of growth might depend nonlinearly on the current population or available resources.

It is worth noting that our postulated solutions tend to show oscillatory tendencies, and such ideas first come from the studies in [10]. Given this behavior, our analysis is tailored for bounded domains. Transitioning from these domains to the broader space $\mathbb{R}^N$ necessitates the incorporation of convergence-based arguments.

Beyond the previously mentioned aspects, there is a valid question we can ask: When trying to analyze the complexity of diffusion phenomena, one might think that introducing a higher order operator related to the fourth derivative might be an oversimplification without a detailed physical connection. In this regard, we can aim to link our problem with a broader spectrum of operators. These operators can involve derivatives of any order, including fractional derivatives, which are commonly used for modeling complex diffusion phenomena. In the realm of fractional partial differential equations, the fractional Laplacian often models anomalous diffusion. Solutions to these equations can show sub-diffusion or super-diffusion, which can look like slowed-down or sped-up oscillations, respectively, especially when observed in wave packet evolutions. Incorporating fractional derivatives can result in solutions with complex spatiotemporal oscillations, different from classical Turing patterns seen in integer-order systems. Higher order operators can introduce more complexity compared to the Laplacian operator, but they still operate within the confines of local interactions. Fractional derivatives, with their non-local nature, can capture the essence of the long-range interactions and memory effects often observed in such anomalous processes [21].

From a numerical and analytical standpoint, both higher order and fractional differential equations present challenges. Higher order equations often require more intricate boundary conditions and can lead to stability concerns in numerical simulations. Fractional equations, given their non-local nature, demand specialized numerical techniques and a careful consideration of their implications [22].

There exist interesting mathematical models involving fractional operators. Their focus ranges from novel modeling of specific diseases using advanced mathematical operators to the study of properties like existence, uniqueness, and stability of solutions in broader contexts. As example, the study [23] presents a model for the AH1N1/09 influenza virus, applying the fractal–fractional operators with two distinct fractal and fractional orders. This model is unique in its utilization of the power law-type kernels. Techniques, including contractions and admissibles and the Leray–Schauder theorem, are employed to study the existence of solutions. A numerical approximation using the Adams–Bashforth scheme is also applied. The model is then translated into a fractional model using the generalized Liouville–Caputo-type operators and subsequently simulated using the Kumar–Erturk method. The analysis in [24] focuses on a p-Laplacian nonperiodic boundary value problem, modeled with generalized Caputo fractional derivatives. The existence and uniqueness of the solutions are established using fixed-point theorems, specifically the Banach and Schauder theorems. The results are also exemplified for practical understanding. The text in [25] proposes a novel approach to modeling fractional multi-term boundary value problems on the graph representation of the glucose molecule. This involves using a new labeling method for the vertices of arbitrary graphs. Existence results are derived with the help of two known fixed-point theorems, and an example is provided for illustration. The analysis in [26] suggests a new fractional model for the human liver using the Caputo–Fabrizio derivative with the exponential kernel. The uniqueness and existence of the model's solution are explored via the Picard–Lindelöf approach and fixed-point theory. Additionally, the homotopy analysis transform method is used for implementation. Comparisons with real clinical data highlight the efficacy of this new fractional model over traditional models. In addition, the study in [27] presents a mathematical model for the transmission dynamics of COVID-19 using the Caputo fractional-order derivative. Key metrics, such as the reproduction number and equilibrium points, are calculated. Fixed-point theory confirms the existence of a unique solution, and the generalized Adams–Bashforth–Moulton method is employed for solving the system. The reference [28] investigates a new mathematical model for HIV using the fractional Caputo–Fabrizio derivative. The existence and uniqueness of the solution are reviewed using fixed-point theory, while the equation itself is tackled with a combination of the Laplace transform and homotopy analysis method. Numerical analytics complement the findings. Further, [29] offers an in-depth look at the emotional and socio-economic implications of the COVID-19 pandemic. The research leverages stochastic mathematical modeling to describe the pandemic dynamics. The stochastic model assesses disease prevalence, existence, and extinction. MATLAB simulations, supported by graphs, offer numerical insights. Eventually, [30] concentrates on the solutions of a nonlinear neutral stochastic fractional differential system, studying their existence and Ulam–Hyers stability. The Banach contraction principle and fixed-point theorems ensure the existence and uniqueness of these solutions. The paper also touches on the fractional calculus' fundamental schemes. In [31], a box model is utilized to investigate hearing loss in children due to the mumps virus using the Caputo–Fabrizio fractional derivative, which considers system memory effects. The research evaluates the basic reproduction number, equilibrium points, and system stability. The existence of a unique solution for the fractional system of the hearing loss model is demonstrated using the Picard–Lindelof method.

The study also explores optimal control strategies for treatment, computes approximate solutions using the Euler method, and assesses the impact of varying fractional-order derivatives and the sensitivity of the basic reproduction number to model parameters. Note that in [32], the study presents a fractal–fractional-order mathematical model for tuberculosis, categorizing the population into six classes, which include susceptible individuals across different age groups, active infected, non-active infected, and recovered individuals. The research defines successive iterative sequences to identify model solutions, investigates uniqueness, and establishes Hyers–Ulam (HU) stability. Numerical solutions are provided using two-step Lagrange polynomials, and the model's applicability and predictions are validated through numerical simulations, taking into account different fractional and fractal orders. In summary, the core themes across these mentioned references [23–32] involve modeling various phenomena, predominantly diseases, using mathematical techniques, primarily involving fractional calculus and fixed-point theory. Each text demonstrates the versatility and applicability of these methods in capturing the complexities of real-world systems and offering novel insights into their behavior.

Upon reviewing the previously mentioned references, it becomes evident that an examination of our problem (1) would be more insightful and versatile if we consider an extended version of the operator in the form:

$$u_t = -\Delta^k u + c \cdot \nabla u^p + u^q,\ \ 0 < k < \infty,\ \ k \neq 1. \tag{2}$$

In this way, we would incorporate more varied and rich discussions that would make our problem an ideal candidate for modeling both non-local and local phenomena. The mathematical nature of fractional operators is notably different compared to higher order operators, making a unified study challenging. Consequently, in what follows, we will focus solely on the analysis related with the higher order operator, leaving the potential correlation of results with fractional operators for future studies.

2 | PRELIMINARY RESULTS

Consider the spatial set $\Omega_r = \{x \in B(0,r) \subset \mathbb{R}^N\}$ and with C^b-boundary ($b \geq 1$).

Definition 1. T is called an extension operator and is defined as

$$T : W^{m,p}(\Omega_r) \to W^{m,p}(\mathbb{R}^N), \tag{3}$$

where $(Tu)(x) = u(x)$ a.e. in Ω_r.

Based on this, the following proposition states:

Proposition 1. *For the extension operator T defined, the following holds:*

$$\|Tu\|_{W^{m,p}(\mathbb{R}^N)} \leq A\|u\|_{W_0^{m,p}(\Omega_r)}, \tag{4}$$

where A is a constant related with Borel measures in partitions of unity.

Proof. Without losing generality, firstly consider that $r \to \infty$ with $p = 2$, then for any $u \in H^m(\Omega_{r\to\infty})$, it holds that

$$(1 + |f|^2)^{\frac{m}{2}} \hat{u}(f) \in L^2(\Omega_{r\to\infty}), \tag{5}$$

where $\hat{u}$ is the Fourier transformation of u and f the Fourier variable. In addition, and for any $n < m - \frac{N}{2}$, the following holds:

$$\|(1 + |f|^2)^{\frac{n}{2}} \hat{u}(f)\|_{L^2(\Omega_{r\to\infty})} \leq A_1 \|Tu\|_{W^{m,p}(\Omega_{r\to\infty})},\ \ A_1 \in \mathbb{R}^+. \tag{6}$$

Using the derivative property of the Fourier transformation for $\partial^\alpha u$ ($\alpha \in \mathbb{N}$) and the last inequality, we can state that each $\partial^\alpha u$ is Hölder continuous and bounded for $|\alpha| < m - \frac{N}{2}$.

Consider now the density property in $W^{m,p}$ spaces. For any $u \in W^{m,p}(\bar{\Omega}_r)$, let us define the following sequence:

$$u_r = \{|\max u(\partial\Omega_r)|, r = 1, 2, 3\ \dots\}. \tag{7}$$

Based on the ordered property in $W^{m,p}$ spaces, we can consider the following expression involving theSPitype="" boundary $\partial\Omega_r$:

$$|\max u(\mathbb{R}^N)| = \lim_{r\to\infty} |\max u(\partial\Omega_r)| < \infty. \tag{8}$$

Considering the previous finite condition for $|\max u(\mathbb{R}^N)|$, along with the condition of a bounded Ω_r with smooth C^b-boundary ($b \geq 1$), we can introduce an approximation technique for any function given in $W^{m,p}(\bar{\Omega}_r)$ by functions in $C^\infty(\mathbb{R}^N)$ and restricted to $\bar{\Omega}_r$. Additionally, for each r, there is a partition of unity that is represented as $\{\delta_i^r\}_{i\in I}$, being I any set of indexes. For any function $u_r \in W^{m,p}(\bar{\Omega}_r)$, it holds that $\delta_i^r u_r \in W^{m,p}(\bar{\Omega}_r)$ and is compactly supported satisfying that $\mathrm{spt}(\delta_i^r u_r) \subset \Omega_r$.

Now, consider the standard mollifier ψ (see [33] for further details). Given any small $\varepsilon_i > 0$, the following is defined:

$$h_i^r := (\delta_i^r u_r) * \psi_{\varepsilon_i} \in C_0^\infty(\bar{\Omega}_r). \tag{9}$$

Considering the limit $r \to \infty$, a partition of unity can be defined as $\{\delta_i\} = \lim_{r\to\infty}\{\delta_i^r\}$. Then, the following function is newly given by

$$h_i := (\delta_i u) * \psi_{\varepsilon_i} \in C_0^\infty(\mathbb{R}^N). \tag{10}$$

Consider now a Borel measure, γ, in each $\{\delta_i^r\}$. If needed, we may introduce any spatial translation such that h_i is located below h_i^r. Then, the following applies:

$$(\gamma(\delta_i)u) * \psi_{\varepsilon_i} \leq (\gamma(\delta_i^r)u_r) * \psi_{\varepsilon_i}, \tag{11}$$

in $\bar{\Omega}_r$ and with $\Omega_{r\to\infty}$.

Now, consider the classical norms in $W_0^{m,p}$ with the density properties of $W_0^{m,p}$ (see [34]) in Ω_r. In addition, we should mention that $\cup_{r=1}^\infty \Omega_r = \mathbb{R}^N$:

$$\|Tu\|_{W^{m,p}(\mathbb{R}^N)} \leq \|u\|_{W^{m,p}(\mathbb{R}^N)} \leq \frac{\gamma(\delta_i^r)}{\gamma(\delta_i)}\|u\|_{W_0^{m,p}(\Omega_r)} = A\|u\|_{W_0^{m,p}(\Omega_r)}, \tag{12}$$

as it was initially stated. □

For the basic equation $u_t = -\Delta^2 u$, the map $V : \mathbb{R}^+ \to \mathbb{R}^+$ with $V(t) = e^{-\Delta^2 t}$ defines a family $\{V(t)\}_{t\in\mathbb{R}^+}$ of bounded linear operators in L^2. The boundedness property of each element of the family can be shown by application of the Plancherel theorem. The family $\{V(t)\}_{t\in\mathbb{R}^+}$ complies with the basic conditions required for the semigroup theory: $V(0) = I,\ V(t+\tau) = V(t)V(\tau),\ V(-t) = (V(t))^{-1}$.

As a consequence, the operator $-\Delta^2$ can be understood as the infinitesimal generator of the family $\{V(t)\}_{t\in\mathbb{R}^+}$ for which the semigroup representation holds. Indeed,

$$-\Delta^2 = \lim_{t\to 0^+}\frac{V(t) - I}{t}, \tag{13}$$

where $I = \lim_{t\to 0^+} V(t) = I$. Furthermore, the coming proposition compiles the applicable semigroup theory conditions:

Proposition 2. *Let $-\Delta^2$ be a linear operator acting on L^2. Consider the family $\{V(t)\}_{t\in\mathbb{R}^+}$ with elements defined by*

$$e^{-\Delta^2 t} := \sum_{k=1}^{\infty}\frac{t^k(-\Delta^2)^k}{k!}, \tag{14}$$

where $(-\Delta^2)^0 = I$. Then, $\{V(t)\}_{t\in\mathbb{R}^+}$ forms a uniformly continuous semigroup in L^2, and the operator $-\Delta^2$ serves as the infinitesimal generator for this semigroup.

Proof. To establish the aforementioned proposition, we draw upon foundational results from semigroup theory as elaborated in [35]. We adapt several of these core principles specifically to the context of the infinitesimal generator $-\Delta^2$:

$$\|V(t)\|_2 = \|e^{-\Delta^2 t}\|_2 \leq \sum_{k=1}^{\infty}\frac{(t\|-\Delta^2\|_2)^k}{k!} = e^{t\|-\Delta^2\|_2}, \quad \forall t > 0. \tag{15}$$

Hence, the family $\{V(t)\}_{t\in\mathbb{R}^+}$ is appropriately defined assuming the operator $(-\Delta^2)$ remains finite within the L^2 norm. An in-depth analysis of the boundedness of solutions under the operator $-\Delta^2$ can be found in Section 2.1. Furthermore, every member of this family adheres to the condition $V(0) = I$, where I denotes the identity operator in L^2.

Now, the uniform continuity is shown as follows:

$$\|\frac{V(t)-I}{t} - (-\Delta^2)\|_2 \leq \frac{1}{t}\sum_{k=2}^{\infty} \frac{t(\|-\Delta^2\|_2)^k}{k!} = \frac{1}{t}(e^{\|-\Delta^2\|_2 t} - I - \|-\Delta^2\|_2\, t), \tag{16}$$

that converges to zero for $t \to 0^+$. Consequently, we affirm that $-\Delta^2$ acts as the infinitesimal generator for the uniformly continuous semigroup family denoted by $\{V(t)\}_{t\in\mathbb{R}^+}$, having its domain specified as $\mathrm{Dom}(-\Delta^2) = L^2$. □

Our subsequent goal is to demonstrate that the family $\{V(t)\}_{t\in\mathbb{R}^+}$ exhibits strong continuity, thereby constituting a C^0-semigroup. To facilitate this, we first introduce the ensuing proposition:

Proposition 3. *Let the family $\{V(t)\}_{t\in\mathbb{R}^+}$ be a C^0-semigroup. Then, there exist constants w and m such that for all $t \geq 0$, the inequality*

$$\|V(t)\|_2 \leq we^{mt} \tag{17}$$

holds.

Proof. First, we established in Proposition 2 that the family $\{V(t)\}$ is uniformly continuous. Consequently, for any suitable $w \geq 1$, the following inequality holds locally for any $t > 0$:

$$\|V(t)\|_2 \leq w. \tag{18}$$

To demonstrate our proposed statement, we employ a proof by contradiction. Suppose, contrary to our claim, that inequality (18) does not hold globally. This presupposes the existence of a decreasing (and converging to zero) time sequence $\{t_n\}$ for which

$$\|V(t_n)\|_2 \geq n, \ \text{as} n \to \infty. \tag{19}$$

Given the semigroup property, $V(t_n)u \to u$ as $n \to \infty$, implying that the sequence $\{V(t_n)u\}$ remains bounded for all $u \in L^2$. Invoking the Banach–Steinhaus theorem, it follows that the family $\{V(t_n)\}$ is bounded. This conclusion is in direct contradiction with our initial supposition in (19). Hence, the original inequality in (17) must hold, particularly noting that $e^{mt} \geq 1$ for all $t \geq 0$, a fact that does not influence the derived inequality. □

The strongly continuous condition follows easily from the last proposition shown:
Consider any $\sigma > 0$ and any function $u \in L^2$, then it holds that

$$\|V(t+\sigma)u - V(t)u\|_2 \leq \|V(t)\|_2\,\|V(\sigma)u - u\|_2 \leq we^{mt}\|V(\sigma)u - u\|_2. \tag{20}$$

Considering $\sigma \to 0^+$, the following holds: $\|V(t+\sigma)u - V(t)u\|_2 \to 0$, leading to conclude that the family of $V(t)$ is formed of strongly continuous elements.

Definition 2. Given a generalized function $u \in H^4(\Omega_r)$, the following weighted norm in defined as

$$\|u\|_\Gamma^2 = \int_{\Omega_r} \Gamma(\omega) \sum_{j=0}^{4} |D^j u(\omega)|^2 d\omega, \ \ \omega \in \Omega_r \subset \mathbb{R}^N \tag{21}$$

where $D^j = \frac{\partial^{|j|}}{\partial\omega_1^{j_1}\partial\omega_2^{j_2}\dots\partial\omega_N^{j_N}}$, with $|j| = \sum_{k=1}^N j_k$, and the multi-index is given as $j = (j_1, j_2, \dots, j_N) \in (\mathbb{N}\cup\{0\})^{\mathbb{N}}$. In addition, the following embedding property follows easily from (21): $u \in H_\Gamma^4(\Omega_r) \subset L_\Gamma^2(\Omega_r) \subset L^2(\Omega_r)$. An expression for the weight Γ is given as follows (see [10, 36]):

$$\Gamma(\omega) = e^{|\omega|^{\frac{4}{3}} - \frac{1}{|\omega|^p}\frac{1}{t^\alpha}\int_0^t \|c\cdot\nabla u(\omega,\tau)\|_2^p d\tau}, \tag{22}$$

where $|\omega| = \sum_{k=1}^{N} |\omega_k|$ and $\alpha > p + 1$.

2.1 Bounding properties in functional spaces

Let us consider the operator $\mathcal{L}$, characterized by diffusion–advection dynamics, given by $\mathcal{L} = -\Delta^2 + pu^{p-1}c \cdot \nabla$. With this in mind, we can now delve into the corresponding problem formulated under this operator:

$$u_t = \mathcal{L}u. \tag{23}$$

The initial distribution in (1) is requested to comply with $u_0 \in W_0^{m,p}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. In the following lemma, further conditions are specified for u_0.

Lemma 1. *Given an initial function u_0 residing in the space $L^2(\mathbb{R}^N)$, the subsequent relationship is valid:*

$$\|u\|_{L^2} \leq \|u_0\|_{L^2}. \tag{24}$$

For any positive real number $d \in \mathbb{R}^+$, and assuming that u_0 lies in both $H^d(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$, we have

$$\|u\|_{H^d} \leq \|u_0\|_{H^d}, \tag{25}$$

and for times t such that $t \geq \frac{d}{4}$:

$$\|u\|_{H^d} \leq \|u_0\|_{L^2}. \tag{26}$$

Additionally, the subsequent bound is established:

$$\|u\|_{\Gamma} \leq \varrho \|u\|_{H^d} \leq \varrho \|u_0\|_{H^d}, \tag{27}$$

where

$$\varrho = 25 \sup\{u, D^1u, D^2u, D^3u, D^4u\}. \tag{28}$$

Proof. A fundamental abstract representation for Equation (23) is given by the following expression:

$$u(x,t) = e^{t\mathcal{L}}u_0(x). \tag{29}$$

Consider the Fourier transformation (f variable in $\mathbb{R}^N$) as per the following expression:

$$\hat{u}(f,t) = e^{t(-|f|^4 + p\hat{u}^{p-1}c \cdot fi)}\,\hat{u}_0(f). \tag{30}$$

Considering now the Plancherel theorem, the following holds:

$$\begin{aligned} \|u\|_{L^2}^2 &= \int_{-\infty}^{\infty} |e^{t2(-|f|^4 + p\hat{u}^{p-1}c \cdot fi)}|\,|\hat{u}_0(f)|^2 df = \int_{-\infty}^{\infty} e^{-2|f|^4 t}|\hat{u}_0(f)|^2 df \\ &\leq \sup_{|f| \in \mathbb{R}} (e^{-2|f|^4 t}) \int_{-\infty}^{\infty} |\hat{u}_0(f)|^2 df = \|u_0\|_{L^2}^2. \end{aligned} \tag{31}$$

Then, $\|u\|_{L^2} \leq \|u_0\|_{L^2}$.

Now, assume the following definition of a mollifying norm for $d \in \mathbb{R}^+$ and $0 \leq t \leq \sigma < \infty$ that satisfies the A_p-condition (refer to [37] for additional insights) for $p = 1$:

$$\|u\|_{H^d}^2 = \int_{-\infty}^{\infty} e^{d|f|^2}\,|\hat{u}(f,t)|^2 df. \tag{32}$$

Then,

$$\|u\|_{H^d}^2 = \int_{-\infty}^{\infty} e^{d|f|^2} |\hat{u}(f,t)|^2 df = \int_{-\infty}^{\infty} e^{d|f|^2} |e^{t2(-|f|^4+p\hat{u}^{p-1}c\cdot fi)}| \, |\hat{u}_0(f)|^2 df$$
$$\leq \sup_{|f|\in\mathbb{R}} (e^{-2|f|^4 t}) \int_{-\infty}^{\infty} e^{d|f|^2} |\hat{u}_0(f)|^2 df = \|u_0\|_{H^d}^2. \tag{33}$$

Consider now that $u_0 \in L^2(\mathbb{R}^N)$, then the following applies:

$$\|u\|_{H^d}^2 = \int_{-\infty}^{\infty} e^{d|f|^2} |\hat{u}(f,t)|^2 df \leq \sup_{|f|\in\mathbb{R}} (e^{d|f|^2} e^{-2|f|^4 t}) \int_{-\infty}^{\infty} |\hat{u}_0(f)|^2 df. \tag{34}$$

By standard operations, it holds that

$$\|u\|_{H^d}^2 \leq \left(\frac{d}{4t}\right)^{1/2} \|u_0\|_{L^2}^2, \quad \|u\|_{H^d} \leq \|u_0\|_{L^2}, \tag{35}$$

for $t \geq \frac{d}{4}$, as initially postulated.

Eventually, the following assessment holds:

$$\|u\|_{\Gamma}^2 = \int_{\Omega_r} \Gamma(f) \sum_{j=0}^{4} |D^j u(f)|^2 df \leq \int_{\Omega_r} e^{d|f|^2} \sum_{j=0}^{4} |D^j u(f)|^2 df \leq \varrho^2 \int_{\Omega_r} e^{d|f|^2} |u(f)|^2 df \leq \varrho^2 \|u\|_{H^d}^2, \tag{36}$$

being $\varrho = 25 \sup_{|f|\in\mathbb{R}} \{u, D^1 u, D^2 u, D^3 u, D^4 u\}$.

Note that the variable ϱ is not precisely assessed, but we can state at least about its boundedness. To this end, the reader is referred to the continuous inclusions for Sobolev spaces discussed in [34, p. 79]. Assume that $u(x,t)$ is differentiable (in the sense of distributions) up to the third spatial order. Hence, the fourth-order spatial derivative in ϱ can be regarded as a controlling variable. If such fourth-order spatial derivative exists, then the mollifying norm given in (32) permits to bound the norm defined in (21). If the fourth-order spatial derivative does not exist, the function $u(x,t)$ is highly irregular and we cannot state on a mollifying norm. □

As previously discussed, the operator $-\Delta^2$ can be understood as the infinitesimal generator of a strongly continuous semigroup under the family $\{V(t)\}_{t\in\mathbb{R}^+}$ formed of bounded linear operators in L^2 given by $V(t) = e^{-\Delta^2 t}$. Consequently, the Duhamel principle can be applied to obtain the following representation of (1):

$$u(t) = e^{-\Delta^2 t} u_0 + \int_0^t \left[c \cdot \nabla (e^{-\Delta^2 (t-\tau) p} u^p(\tau)) + e^{-\Delta^2 (t-\tau)} u^q(\tau) \right] d\tau. \tag{37}$$

Now, consider the fundamental problem $u_t = -\Delta^2 u$ with a Dirac pulse $u(x,0) = \delta(x)$. A solution to this problem is given under the Fourier representation as follows:

$$\hat{u}(t) = e^{-|f|^4 t} \hat{u}_0(f). \tag{38}$$

A kernel for the fundamental problem can be obtained as

$$\kappa(x,t) = F^{-1}(e^{-|f|^4 t}) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-|f|^4 t - if\cdot x} df = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-|f|^4 t} cos(f \cdot x) df. \tag{39}$$

The right-hand integral is bounded for any $f \in \mathbb{R}^N$. Consequently, the abstract representation given in (37) can be expressed in terms of the kernel. First, the following operator in $H^4_\Gamma(\Omega_r)$ is defined as

$$G_{u_0,t} : H^4_\Gamma(\Omega_r) \to H^4_\Gamma(\Omega_r), \tag{40}$$

such that the expression (37) is rewritten as

$$G_{u_0,t}(u) = \kappa(x,t) * u_0(x) + \int_0^t \left[c \cdot \nabla \kappa(x,t-\tau) * u^p(\tau) + \kappa(x,t-\tau) * u^q(x,\tau) \right] d\tau, \tag{41}$$

where "$*$" refers to the spatial convolution operator and t shall be understood as a positive parameter. Note that the following assessment has been implicitly done in the previous expression:

$$\begin{aligned}\kappa(x,t) * c\cdot\nabla u^p(x,\tau) &= \int_{-\infty}^{\infty}\kappa(x-\alpha,t)c\cdot\nabla u^p(\alpha,\tau)\,d\alpha = -\int_{-\infty}^{\infty}u^p(\alpha,\tau)c\cdot\nabla\kappa(x-\alpha,t)\,d\alpha\\ &= -\int_{-\infty}^{\infty}u^p(\alpha,\tau)c\cdot\nabla_{(x-\alpha)}\kappa(x-\alpha,t)\frac{\partial(x-\alpha)}{\partial\alpha}d\alpha = \int_{-\infty}^{\infty}u^p(\alpha,\tau)c\cdot\nabla_{(x-\alpha)}\kappa(x-\alpha,t)\\ &= c\cdot\nabla\kappa(x,t) * u^p(x,t).\end{aligned} \tag{42}$$

The coming intention consists in proving the boundedness of the t-parametric operator $G_{u_0,t}$.

Lemma 2. *The single t-parametric operator $G_{u_0,t}$ is bounded in $H^4_\Gamma(\Omega_r)$ under the norm (21). The extension operator $TG_{u_0,t}$ (refer to Definition 1) is bounded in the space $H^4(\mathbb{R}^N)$.*

Proof. Firstly of all, we shall show that

$$A_0\|u_0\|_\Gamma \le \|u\|_\Gamma, \tag{43}$$

where the constant A_0 shall be further specified. Indeed,

$$\begin{aligned}\|u\|_\Gamma^2 &= \int_{\Omega_r}\Gamma(f)\sum_{j=0}^{4}|D^j\hat{u}(f)|^2df = \int_{\Omega_r}\Gamma(f)\sum_{j=0}^{4}|D^j\left[e^{t(-|f|^4+p\hat{u}^{p-1}c\cdot fi)}\hat{u}_0\right]|^2df\\ &\ge \int_{\Omega_r}\Gamma(f)\sum_{j=0}^{4}|D^j\left[e^{t(-|f|^4+p\hat{u}^{p-1}c\cdot fi)}\right]|^2\sum_{j=0}^{4}|D^j\hat{u}_0|^2df \ge A_0^2\int_{\Omega_r}\Gamma(f)\sum_{j=0}^{4}|D^j\hat{u}_0|^2df = A_0^2\|u_0\|_f^2,\end{aligned} \tag{44}$$

such that

$$A_0^2 = \inf_{f\in B_r}\left\{\sum_{j=0}^{4}|D^j\left[e^{t(-|f|^4+p\hat{w}^{p-1}c\cdot fi)}\right]|^2\right\} > 0, \tag{45}$$

and small in $B_r = \{f,\ |f| < r\}$, for $r > 0$.

Now, returning to the expression for $G_{u_0,t}$ in (41):

$$\begin{aligned}\|G_{u_0,t}(u)\|_\Gamma \le \|G_{u_0,t}\|_\Gamma\|u\|_\Gamma &\le \|\kappa\|_\Gamma\|u_0\|_\Gamma + \int_0^t\left[\|c\cdot\nabla\kappa\|_\Gamma\|u^p\|_\Gamma + \|\kappa\|_\Gamma\|u^q\|_\Gamma\right]d\tau\\ &\le \left[\|\kappa\|_\Gamma\frac{1}{A_0t} + \int_0^t\left[\|c\cdot\nabla\kappa\|_\Gamma\|u_0^{p-1}\|_{H^d} + \|\kappa\|_\Gamma\|u_0^{1-q}\|_\Gamma\right]d\tau\right]t\|u\|_\Gamma.\end{aligned} \tag{46}$$

We should note that the inequalities (33) and (36) have been used as follows (see for the term $\|u^{p-1}\|_\Gamma$):

$$\|u^{p-1}\|_\Gamma \le \|u^{p-1}\|_{H^d} \le \|u_0^{p-1}\|_{H^d}. \tag{47}$$

Then, it holds that

$$\|G_{u_0,t}\|_\Gamma \le \left[\|\kappa\|_\Gamma\frac{1}{A_0t} + \int_0^t\left[\|c\cdot\nabla\kappa\|_\Gamma\|u_0^{p-1}\|_{H^d} + \|\kappa\|_\Gamma\|u_0^{1-q}\|_\Gamma\right]d\tau\right]t < \infty, \tag{48}$$

for any t-parameter in $0 < t < \infty$. Furthermore, it should be noted that the right-hand expression is bounded locally for any t-value. The action of the operator $G_{u_0,t}$ can be extended to the whole space $\mathbb{R}^N$ by the application of the extension operator given in Proposition 1 and the relation $\|G_{u_0,t}\|_{H_0^4(\Omega_r)} \leq \|G_{u_0,t}\|_\Gamma$, a.e. in Ω_r. As a consequence, it holds that

$$\|TG_{u_0,t}\|_{H^4(\mathbb{R}^N)} \leq A\|G_{u_0,t}\|_{H_0^4(\Omega_r)} \leq A\|G_{u_0,t}\|_\Gamma. \tag{49}$$

□

The coming objective is to provide an analysis to support the existence of solution for the problem (1) in $H_0^4(\Omega_r)$.

Theorem 1. *Based on the representations in (37) and (41) and a fixed-point argument, assume that $u(t) = G_{u_0,t}(u(t))$, for $0 < t \leq \sigma$. Then, there exist weak solutions to the problem (1) complying with $u \in L^2(0, \sigma; H_0^4(\Omega_r))$.*

Proof. Consider the sequence of open subsets Ω_r as introduced in Section 2 and the space $H_0^4(\Omega_r)$. There exists a set of eigenfunctions obtained from a reproducing kernel forming an orthonormal basis (for further insights about the orthonormal conditions in functional spaces, the reader is referred to [38]). Hence, assume that $\{\psi_j(x)\}$, $j = 1, 2, \ldots$ represents an orthogonal basis for the space $H_0^4(\Omega_r)$. Given a sequence of solutions to (1) by $\{u_n\}$, the following spanned expression holds:

$$u_n(x,t) = \sum_{j=1}^{n} h_{n,j}(t)\psi_j(x), \; n = 1, 2, 3 \ldots . \tag{50}$$

Note that each u_n complies with the following ordinary differential equation:

$$(u_{n,t}, \psi_j) - (\nabla(\nabla \cdot \nabla u_n), \nabla\psi_j) + (cu_n^p, \nabla\psi_j) - (u_n^q, \psi_j) = 0, \tag{51}$$

where $(\cdot, \cdot)$ is the inner in the space H^4.

Now, the initial conditions for the elements in the sequence are spanned as

$$u_n(x,0) = \sum_{j=1}^{n} h_{n,j}(0)\psi_j(x), \; n = 1, 2, 3 \ldots , \tag{52}$$

where $h_{n,j}(0) = (u_n(x,0), \psi_j(x))$ are constants.

Consider now the Peano theorem from the basic theory of ordinary differential equations. The analysis about existence of general solutions, complying with $u \in L^2(0, \sigma; H_0^4(\Omega_r))$, requires preliminary to consider that

$$h_{n,j}(t) \in C^1([0, \sigma]; H_0^4(\Omega_r)), \tag{53}$$

so that $u_n \in C^1([0, \sigma]; H_0^4(\Omega_r))$.

The objective is to determine estimates for each element of the sequence $u_n(t)$. First of all, take Equation (51) and make the multiplication by $h_{n,j}(t)$. Making the sum $j = 1$ to $j = n$, the following holds:

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_2^2 + \|\nabla \cdot \nabla u_n\|_2^2 + \|cu_n^p\|_2\|\nabla u_n\|_2 \leq \|u_n^q\|_2 \leq \|u_n\|_2^2,$$

at least in a region where $u \geq 1$. Given the space $W^{m,p}(\Omega_r)$, define $\alpha_1 = \text{int}\{m - \frac{N}{p}\}$, then the following continuous inclusion applies (see [34, p. 79]):

$$W^{m,p}(\Omega_r) \hookrightarrow C^{\alpha_1}(\Omega_r). \tag{54}$$

In the current study, any solution admits derivatives (in a distributional sense) up to the fourth order, then $m = 4$ with $p = 2$ and $\alpha_1 = \text{int}\{4 - \frac{N}{2}\}$. Given the continuous embedding (54) and the requirement of a compact support for each $\psi_j(x) \in H_0^4(\Omega_r)$, we can postulate that the terms $\|\nabla \cdot \nabla u_n\|_2^2$ and $\|cu_n^p\|_2\|\nabla u_n\|_2$ are bounded in Ω_r. Consequently, the inequality (2.1) can be expressed as

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_2^2 \leq \|u_n\|_2^2. \tag{55}$$

The following estimate holds for each element: $\|u_n\|_2^2 \leq e^{2t},\ 0 < t \leq \sigma$.

Consider now the integration of (55) between $(0, \sigma]$:

$$\frac{1}{2}\|u_n\|_2^2 - \frac{1}{2}\|u_n(0)\|_2^2 \leq \int_0^\sigma \|u_n\|_2^2 \leq \frac{1}{2}e^{2\sigma}, \tag{56}$$

which leads to state that each element of the sequence, u_n, is globally bounded in $0 < t \leq \sigma$.

Consider now the bound obtained in (56), then we can postulate that there exists a function $u \in L^2(0, \sigma; H_0^4(\Omega_r))$ and a subsequence $\{u_n\},\ n = 1, 2, 3 \ldots$ such that for any $t \in (0, \sigma]$ and for $n \to \infty$, the following holds:

$$u_n \to u, \tag{57}$$

in a weak approach in $L^2(0, \sigma; H_0^4(\Omega_r))$.

Again take into account the bound in (56), the mentioned weak converge of $\{u_n\}$ together with the Aubin–Lions–Dubinskii compactness theorem (see [39]). Then, $u_n \to u$ strongly in $C(0, \sigma; L^2(\Omega_r))$.

Recall that the Aubin–Lions–Dubinskii theorem requires that the derivative $\frac{\partial u_n}{\partial t}$ shall be bounded in a Banach space. This can be shown based on the expression (55) in $0 < t \leq \sigma$.

Once we have shown the strong convergence in $\{u_n\}$, the following expression represents the problem (1):

$$(u_t, \psi_j) - (\nabla(\nabla \cdot \nabla u), \nabla\psi_j) + (cu^p, \nabla\psi_j) - (u^q, \psi_j) = 0,\ \forall j. \tag{58}$$

Consider a function Ψ given as a combination of $\{\psi_j\}$, then

$$(u_t, \Psi) - (\nabla(\nabla \cdot \nabla u), \nabla\Psi) + (cu^p, \nabla\Psi) - (u^q, \Psi) = 0. \tag{59}$$

This last equation, together with $u_0(x) \in W_0^{m,p}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ (recall that $m = 4,\ p = 2$) and with the convergence in the mentioned dense spaces for any arbitrary initial distribution, allows us to state that Equation (1) has weak solutions in $0 < t < \sigma$ where $\sigma < \infty$. □

We should mention that in Lemma 2, we proved the global bound of solutions based on the boundedness of the t-parametric operator $G_{u_0,t}$. Additionally, in Theorem 1, we showed the existence of solutions based on the convergence argument in dense spaces. Note that the convergence approach is valid for any $\sigma \gg 1$ while preserving the global bound of solutions. Then, the existence of solutions analysis has been provided in a global sense and for any $0 < t < \sigma < \infty$. Furthermore, the results apply to $\mathbb{R}^N$. To show this, it suffices to consider the extension operator as per Definition 1: $\|TG_{u_0,t}\|_{H^4(\mathbb{R}^N)} \leq A\|G_{u_0,t}\|_{H_0^4(\Omega_r)}$.

Based on the rationales exposed, we can conclude that there exist global solutions $u(x, t)$ for $(x, t) \in \mathbb{R}^N \times (0, \sigma)$ to the problem (1).

2.2 | Uniqueness of solutions analysis

The analysis about uniqueness of solutions is based on a fixed-point argument given by the operator $G_{u_0,t}$ in (40) understood as a mapping of the form: $u(x, t) = G_{u_0,t}(u(x, t)),\ (x, t) \in \Omega_r \times (0, \sigma]$. To start, assume that there exist two solutions $u_1(x, t)$ and $u_2(x, t)$ satisfying the expression (41) and such that both solutions have the same initial distribution $u_0(x)$. Making the difference between $G_{u_0,t}(u_1) - G_{u_0,t}(u_2)$ in $t \in (0, \sigma]$ and considering the norm (21), the following holds:

$$\|G_{u_0,t}(u_1) - G_{u_0,t}(u_2)\|_\Gamma \le \int_0^t \|c \cdot \nabla\kappa(x,t-\tau) * (u_1^p - u_2^p) + \kappa(x,t-\tau) * [u_1^q - u_2^q]\|_\Gamma d\tau$$
$$= \int_0^t \| \int_t^\tau \{c \cdot \nabla\kappa(x,t-\tau-r)(u_1^p - u_2^p) + \kappa(x,t-\tau-r)[u_1^q - u_2^q]\} dr\|_\Gamma d\tau$$
$$\le \int_0^t \int_t^\tau \{\|c \cdot \nabla\kappa(x,t-\tau-r)(u_1^p - u_2^p)\|_\Gamma + \|\kappa(x,t-\tau-r)[u_1^q - u_2^q]\|_\Gamma\} dr\, d\tau \tag{60}$$
$$= \int_0^t \int_t^\tau \{\|c \cdot \nabla\kappa(x,t-\tau-r)\|_\Gamma \|u_1^p - u_2^p\|_\Gamma + \|\kappa(x,t-\tau-r)\|_\Gamma \|u_1^q - u_2^q\|_\Gamma\} dr\, d\tau$$
$$\le N_1 \int_0^t \int_t^\tau \{\|u_1^p - u_2^p\|_\Gamma + \|u_1^q - u_2^q\|_\Gamma\} dr\, d\tau.$$

Note that κ and $\nabla\kappa$ are bounded functions (see the form of 39), as a consequence, it holds that

$$N_1 = \sup\{\|\kappa(x,t-\tau-r)\|_\Gamma, \|c \cdot \nabla\kappa(x,t-\tau-r)\|_\Gamma;\ \forall t \in (0,\sigma],\ x \in \Omega_r\}, \tag{61}$$

and for any $\tau > 0$, $r > 0$.

For the assessments of the integrals given in (60), consider the regular functions:

$$A(\epsilon,\tau) = \begin{bmatrix} \frac{u_1(\epsilon,\tau)^p - u_2(\epsilon,\tau)^p}{u_1(\epsilon,\tau)-u_2(\epsilon,\tau)} \text{ for } u_1 \not\equiv u_2 \\ p u_1^{p-1} \text{ otherwise} \end{bmatrix},\ B(\epsilon,\tau) = \begin{bmatrix} \frac{u_1(\epsilon,\tau)^q - u_2(\epsilon,\tau)^q}{u_1(\epsilon,\tau)-u_2(\epsilon,\tau)} \text{ for } u_1 \not\equiv u_2 \\ \delta^{q-1} u_1 \text{ otherwise} \end{bmatrix}, \tag{62}$$

for $0 < \delta \ll 1$. Given a fixed value of ε and with $\tau = \sigma$, the last two expressions are bounded and satisfy that $0 \le A(\epsilon,\tau) \le C_0(p, \|u_0\|_\infty, \sigma)$, $0 \le B(\epsilon,\tau) \le C_1(q, \|u_0\|_\infty, \sigma)$. Then, the following bounds hold:

$$\|u_1^p - u_2^p\|_\Gamma \le C_0^* \|u_1 - u_2\|_\Gamma,\quad \|u_1^q - u_2^q\|_\Gamma \le C_1^* \|u_1 - u_2\|_\Gamma, \tag{63}$$

where $C_0^* = \|C_0\|_\Gamma$ and $C_1^* = \|C_1\|_\Gamma$.

Coming to (60), the following holds:

$$\|G_{u_0,t}(u_1) - G_{u_0,t}(u_2)\|_\Gamma \le N_1(C_0^* + C_1^*) \int_0^t \int_t^\tau \|u_1 - u_2\|_\Gamma d\tau dr = N_1(C_0^* + C_1^*) t(t-\tau) \|u_1 - u_2\|_\Gamma. \tag{64}$$

Given the time interval $0 < t < \tau \le \sigma$, the uniqueness of the solutions is shown in the case of $u_1 \swarrow u_2$ that leads to a contractive map $G_{u_0,t}$ complying with $G_{u_0,t}(u_1) \swarrow u_1$ in the functional space H_Γ^4.

The obtained result concerning the uniqueness of solutions applies to the solutions in the whole domain $\mathbb{R}^N$. To show this, it suffices to apply the extension operator given in Definition 1 together with: $\|T G_{u_0,t}\|_{H^4(\mathbb{R}^N)} \le A \|G_{u_0,t}\|_{H_0^4(\Omega_r)}$.

3 | PROFILES OF ANALYTICAL SOLUTIONS

The profiles of analytical solutions are explored based on an exponential point scaling given by

$$u = e^v. \tag{65}$$

The reader is referred to the references [40–43] wherein the formalism of the exponential scaling, together with applications in physics, are further discussed.

The exponent function v is in general a complex map (refer to [10]): $v : X \times [0,\sigma] \to \mathbb{C}$.

The function v complies with a Hamilton–Jacobi equation (see [44]):

$$v_t = H_4(v, \nabla v) + P_4\left(v, \frac{\partial^{|j|} v}{\partial x_1^{j_1} \partial x_2^{j_2} \dots \partial x_N^{j_N}}\right),\quad |j| = \sum_{i=1}^N j_i,\ \ j = (j_1, j_2, \dots, j_N) \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}, \tag{66}$$

where

$$H_4(v) = -(\nabla v)^2(\nabla v)^2 + cp\nabla v e^{(p-1)v} + e^{qv}. \tag{67}$$

In addition,

$$P_4(v) = -\Delta^2 v - \Delta(\nabla v \cdot \nabla v) - 2\nabla v \cdot \nabla\Delta v - 2(\nabla v \cdot \nabla v)\Delta v - 2\nabla v \cdot \nabla(\nabla v \cdot \nabla v) - (\Delta v)^2. \tag{68}$$

The algebraic order for P_4 is three. The operator H_4, of Hamilton–Jacobi type, is of order 4. The mentioned orders can be made explicit by standard operations involving a sufficiently smooth function $\Theta \in C_0^\infty$, such that given any $\Xi \in \mathbb{R}^+$, the following holds:

$$|P_4(\Xi\Theta)| = O(\Xi^3) < H_4(\Xi\Theta) = O(\Xi^4). \tag{69}$$

After making the balance among the orders in the expression (66), we only consider the leading terms, so that the new equation is given by

$$v_t = -(\nabla v)^2(\nabla v)^2 + cp\nabla v e^{(p-1)v} + e^{qv}. \tag{70}$$

The intention is to explore solutions to the last Equation (70). First, consider that any solution is expressed under separated variables (see [44] for additional insights):

$$v(x,t) = (\varphi + t)^{-\frac{1}{3}}\Lambda(x), \tag{71}$$

where $\varphi < t \leq \sigma$.

Making the substitution in (70) and considering an asymptotic approximation given by $\varphi \gg 1$:

$$-\frac{1}{3}\Lambda = (\nabla\Lambda)^4 + cp(\varphi + t)\nabla\Lambda + e^{q(\varphi+t)^{-\frac{1}{3}}\Lambda(x)}. \tag{72}$$

The idea is to study the behavior of an asymptotic solution. Therefore, it holds that

$$e^v = e^{(\varphi+t)^{-\frac{1}{3}}\Gamma(x)} \to 1, \tag{73}$$

for $t \to \infty$.

Consider now the balance for $|\Lambda_x| \ll 1$ in (72). The leading terms are now given by

$$-\frac{1}{3}\Lambda = cp(\varphi + t)\nabla\Lambda + a, \tag{74}$$

where $a \to 1$ in the asymptotic approach $t \to \infty$. Hence, after the resolution of (74),

$$\Gamma(x) = 3\left(e^{-\frac{x}{3|c|p(\varphi+t)}} - a\right), \tag{75}$$

where $|c| = \sum_{j=1}^{N} |c_j|$.

Again, under the same asymptotic approach $t \to \sigma \gg 1$, the following moving front is obtained:

$$|x| = 3|c|p\ln(a)t. \tag{76}$$

Now, we make the balance in the advection term: $O(\nabla\Gamma) < O(\varphi + t)$. Considering the asymptotic approximation $|x| \gg 1$:

$$-\frac{1}{3}\Gamma = (\nabla\Gamma)^4 + a. \tag{77}$$

A solution for this last equation reads:

$$\Gamma(x) = 3\left(\frac{1}{4}H(i)|x|\right)^{\frac{4}{3}} - 3a, \tag{78}$$

being $H(i) = (-1)^{\frac{1}{4}}$ and i the imaginary unit. Eventually, a profile of solution in the asymptotic approximation is given as follows:

$$v(x,t) = t^{-\frac{1}{3}}\, 3\left(\left(\frac{1}{4}H(i)|x|\right)^{\frac{4}{3}} - a\right). \tag{79}$$

Now, in return to the single scaling (65), the following profile of solution to (1) holds for $t \to \varphi \gg 1$:

$$u(x,t) = e^{-3at^{-\frac{1}{3}}}\, e^{3t^{-\frac{1}{3}}\left(\frac{1}{4}H(i)|x|\right)^{\frac{4}{3}}}, \tag{80}$$

acting in the wave front given by (76). We remark the oscillating behavior of the profile obtained given the complex number $H(i)$. This oscillating condition is a well-known behavior of solutions to higher order operators (see the reference [10]) and will be further characterized in the numerical assessment introduced in the coming section.

4 | NUMERICAL ASSESSMENTS

Once we have obtained an analytical solution in (80) as an asymptotic approach, the intention now to introduce a numerical assessment for (1), so that the asymptotic approach (and the hypothesis) made can be validated. To this end, we make use of the bvp4c code in Matlab. The considered initial condition shall comply with $u_0(x) \in H_0^4(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$. With no loss of generality and in the sake of simplicity, we consider that $u_0(x) = 1 - k_s\,|x|^4$ where k_s is a suitable constant that is implicitly assessed within the numerical assessments. We should mention that the bvp4c code is constructed based on a Runge–Kutta implicit approach with interpolant extensions (refer to [45] for further insights). The boundary condition, required by the collocation method used by the bvp4c code, has been assumed as: $u(|x| \gg 1) = 0$.

The integration domain is sufficiently large to avoid the impact of the boundary conditions over the solutions. Hence, $|x| \in (-5000{,}5000)$. The number of nodes considered is of 100000 with an absolute error for convergence purposes of 10^{-5}.

The results are provided locally in time and the spatial variable is represented by: $|x| = \sum_{j=1}^{3} |x_j|$. Based on the results obtained, the asymptotic solution profile in (80) fits close with the numerical solution of the problem (1) for $|x| \gg 1$, as it was considered for the derivation of Equation (77). With the intention of providing an idea about the closeness of

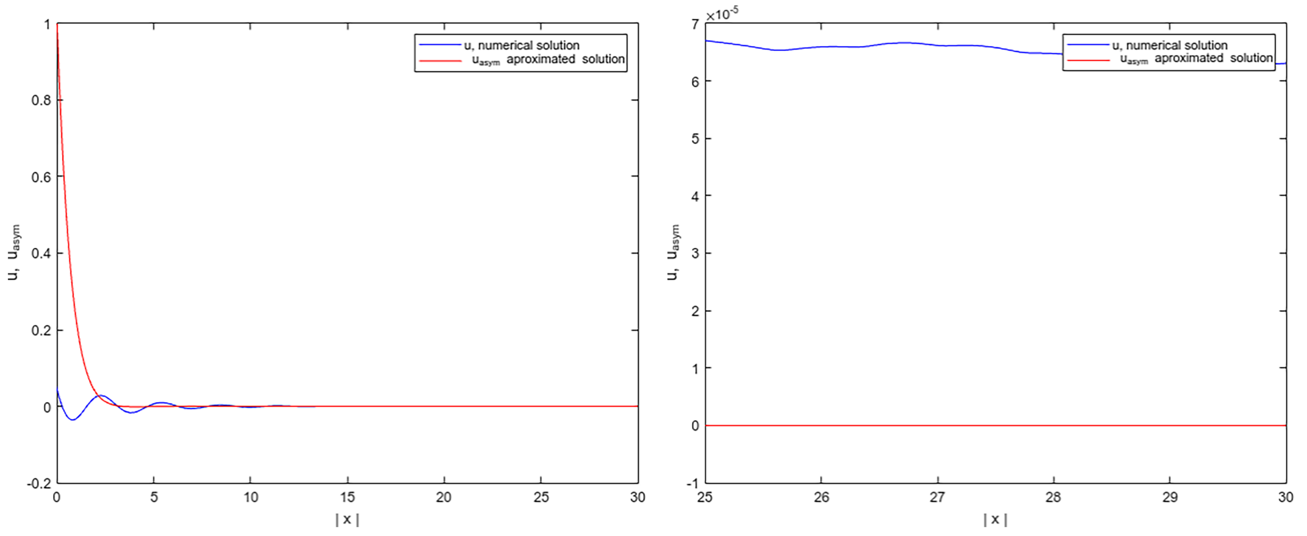

FIGURE 1 Solutions for $t = 500$ (left) and with a zooming (right). The value of $t = 500$ was selected after several trials and was sufficient so as to assume an asymptotic approach $t \gg 1$. The solutions have been obtained for $p = 2$ and $q = 0.5$ in (1). For $|x| > 25.378$, the global error between the asymptotic solution and the numerical is $\leq 10^{-4}$. For other time level, such spatial value varies. Note the oscillatory behavior of both solutions. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 For each time level and for $x \geq x_m$, the global distance between the analytical solution (80) and the numerical one to the problem (1) is $\leq 10^{-4}$, this is $|u_{num} - u_{asym}| \leq 10^{-4}$.

Time level	x_m
1000	27.582
10,000	35.247
100,000	59.278
1,000,000	89.968

solutions, the numerical and the analytical solutions are considered as close whenever the global distance (in the sense of error) complies with $|u_{num} - u_{asym}| \leq 10^{-4}$ for any $|x| \geq x_m$ (see the Figure 1 and the Table 1 for the assessed values of x_m).

The numerical assessments have been executed for different time levels and the a value for x_m has been provided (see the Table 1)

5 | CONCLUSIONS

The problem given in (1) and posed in $\mathbb{R}^N$, $(N > 1)$ has been studied based on analytical and numerical conceptions. The higher order operator $-\Delta^2$ has been proved as the infinitesimal generator of a strongly continuous semigroup of the form $e^{-\Delta^2 t}$. As a consequence, the analysis related with the regularity, existence, and uniqueness of solutions has been formulated with an abstract representation based on the semigroup. To complete our assessments, we have obtained profiles of asymptotic solutions. Such an analysis was supported by a single-point exponential scaling that led to a Hamilton–Jacobi equation. This last equation was solved making use of a separation of variables technique. Finally, a numerical exercise was introduced based on the bvp4c code in Matlab. The numerical approach has allowed us to confirm the validity of the assumptions done during the asymptotic analysis of solutions. Indeed, the asymptotic solution and the numerical one are sufficiently close under the region of validity of the asymptotic approximation.

AUTHOR CONTRIBUTION

José Luis Díaz Palencia: Conceptualization; investigation; writing—original draft; methodology; validation; visualization; writing—review and editing; software; formal analysis; data curation.

CONFLICT OF INTEREST STATEMENT

The author states that there is no conflict of interest.

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