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About the existence of solutions in L^2 spaces for a multidimensional incompressible flow under the $k - \varepsilon$ turbulence model

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Abstract
This study examines the multidimensional Navier–Stokes equations in conjunction with the $k-\varepsilon$ turbulence model to establish the existence of weak solutions for the velocity vector, turbulent kinetic energy, and dissipation rate, denoted as $(\overline{\mathbf{u}}, k, \varepsilon)$. Under certain conditions in external forces, initial, and boundary data, we prove that weak solutions exist within the L^2 space. The provided result implies the existence and boundedness of solutions, which can be of help for further computational investigations under the scope of the $k - \varepsilon$ turbulence model.

Keywords Navier–Stokes equations · Turbulence · $k - \varepsilon$ model · Weak solutions · Hydrodynamics · Regularity of solutions.

1 Introduction

The Navier–Stokes equations govern the motion of fluid substances in areas like aerodynamics, meteorology or marine engineering. Among these, the three-dimensional incompressible Navier–Stokes equations stand as one of the most challenging open problems in mathematical physics, particularly concerning the existence and smoothness of their solutions [1, 2].

In two dimensions, the theory is more mature; Ladyzhenskaya [1] and others have established the global existence and uniqueness of strong solutions. This progress is largely attributable to the ability to bound the vorticity, thereby controlling solution regularity. However, extending these results to three dimensions introduces complexities, primarily due to the nonlinear advection term that governs the transport of vorticity. This nonlinearity exacerbates the interaction between vorticity and velocity, making the analysis considerably more complex.

Classical approaches to understanding the regularity of solutions in multidimensional Navier–Stokes equations have significantly advanced the field. Leray [2] introduced the concept of weak solutions for the three-dimensional case, proving their global existence. Nevertheless, the regularity and uniqueness of these solutions over time remain unresolved. Subsequent research has focused on identifying regularity criteria; Prodi [3] and Serrin [4] developed conditions based on L^p norms of velocity fields, suggesting that solutions remain regular provided these norms do not blow up within finite time.

Recent contributions by Terence Tao [5–7] have further illuminated aspects of multidimensional Navier–Stokes solutions, exploring fluid dynamics in potential fields and on compact manifolds. Chae and Choe [8] examined vorticity conditions in $\mathbb{R}^3$, providing criteria for the global continuation of strong solutions. Miller [9] investigated global existence under nearly two-dimensional initial data and connected two and three-dimensional dynamics. Additionally, works by Foias [10], Danchin [11], Dong and Gu [12] and Diaz [13] have explored Gevrey class regularity and partial regularity in higher dimensions, respectively.

Despite these advancements, the coupling of Navier–Stokes equations with turbulence models, such as the $k - \varepsilon$ model, remains less explored in terms of explicit and formal results regarding the regularity of solutions. This paper presents the existence of weak solutions for the coupled system comprising the Navier–Stokes equations and the $k - \varepsilon$ turbulence model. Theorem 1 asserts that under appropriate conditions, including external forces and Neumann boundary conditions, there exist weak solutions $(\overline{\mathbf{u}}, k, \varepsilon)$ that reside in $L^2(\Omega)$. We introduce a coupled analysis of velocity, turbulent kinetic energy, and dissipation rate.

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The remainder of this paper is structured as follows. Section 2 details the Navier–Stokes equations and the $k - \varepsilon$ turbulence model, along with the weak formulation and boundary conditions. Section 3 presents Theorem 1 and provides a proof utilizing Galerkin approximations and fundamental functional analysis theorems. We acknowledge that theorems employed in this proof (for example the Banach-Alaoglu Theorem and the Aubin-Lions Lemma) are well-established in the existing mathematical literature. Consequently, to maintain clarity and conciseness, we present only the essential ideas and, where necessary, briefly recall pertinent lemmas for simplicity. For further details of these standard results, we refer the reader to the relevant references in the literature, particularly [14–16]. Finally, Sect. 4 summarizes the findings.

2 The Navier–Stokes equations coupled with the $k - \varepsilon$ turbulence model

Consider an incompressible fluid flow within a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) with a sufficiently smooth boundary $\partial\Omega$. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ denote the velocity field, and p the pressure. The incompressible Navier–Stokes equations are given by:

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}. \tag{2}$$

Here:

- $\rho > 0$ is the constant fluid density,
- $\mu > 0$ is the dynamic viscosity,
- $\mathbf{F}$ represents external body forces (e.g., gravity).

In turbulent flows, the velocity and pressure fields exhibit both mean and fluctuating components due to the chaotic and irregular nature of turbulence. Reynolds decomposition separates these instantaneous quantities into their mean and fluctuating parts as follows:

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \tag{3}$$

$$p = \overline{p} + p', \tag{4}$$

where:

- $\overline{\mathbf{u}}$ denotes the time-averaged (mean) velocity field,
- $\mathbf{u}'$ represents the fluctuating (turbulent) velocity component,
- $\overline{p}$ and p' are the mean and fluctuating pressure components, respectively.

The decomposition relies on the following well-known assumptions:

1. **Linearity of Averaging:** The averaging operator is linear, i.e., for any two functions f and g, and constants a and b,

   $$\overline{af + bg} = a\overline{f} + b\overline{g}.$$

2. **Mean of Fluctuations:**

   $$\overline{\mathbf{u}'} = \mathbf{0}, \quad \overline{p'} = 0.$$

   This implies that the mean of the fluctuating components is zero.

These assumptions ensure that the mean and fluctuating components are orthogonal in the averaging process, and this facilitates the derivation of the RANS equations.

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\overline{\mathbf{u}} + \mathbf{u}') = \nabla \cdot \overline{\mathbf{u}} + \nabla \cdot \mathbf{u}' = 0.$$

Taking the time average of both sides:

$$\overline{\nabla \cdot \overline{\mathbf{u}} + \nabla \cdot \mathbf{u}'} = \nabla \cdot \overline{\overline{\mathbf{u}}} + \overline{\nabla \cdot \mathbf{u}'} = 0.$$

Since $\overline{\mathbf{u}}$ is the mean velocity field and does not fluctuate:

$$\overline{\overline{\mathbf{u}}} = \overline{\mathbf{u}},$$

and by assumption:

$$\overline{\mathbf{u}'} = \mathbf{0}.$$

Assuming sufficient smoothness and linearity of differentiation and averaging:

$$\overline{\nabla \cdot \mathbf{u}'} = \nabla \cdot \overline{\mathbf{u}'} = \nabla \cdot \mathbf{0} = 0.$$

Therefore, the averaged continuity equation simplifies to:

$$\nabla \cdot \overline{\mathbf{u}} = 0. \tag{5}$$

Substituting the Reynolds decomposition into the momentum equation (2):

$$\rho \left(\frac{\partial (\overline{\mathbf{u}} + \mathbf{u}')}{\partial t} + (\overline{\mathbf{u}} + \mathbf{u}') \cdot \nabla (\overline{\mathbf{u}} + \mathbf{u}') \right)$$
$$= -\nabla(\overline{p} + p') + \mu \nabla^2 (\overline{\mathbf{u}} + \mathbf{u}') + \mathbf{F}.$$

Expanding the convective term:

$$(\overline{\mathbf{u}} + \mathbf{u}') \cdot \nabla (\overline{\mathbf{u}} + \mathbf{u}') = \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \overline{\mathbf{u}} + \mathbf{u}' \cdot \nabla \mathbf{u}'.$$

Substituting back into the momentum equation:

$$\rho\left(\frac{\partial\overline{\mathbf{u}}}{\partial t}+\frac{\partial\mathbf{u}'}{\partial t}+\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}}+\overline{\mathbf{u}}\cdot\nabla\mathbf{u}'+\mathbf{u}'\cdot\nabla\overline{\mathbf{u}}+\mathbf{u}'\cdot\nabla\mathbf{u}'\right)$$
$$=-\nabla\overline{p}-\nabla p'+\mu\nabla^2\overline{\mathbf{u}}+\mu\nabla^2\mathbf{u}'+\mathbf{F}.$$

Applying the time averaging operator $\bar{\cdot}$ to both sides of the expanded momentum equation:

$$\overline{\rho\left(\frac{\partial\overline{\mathbf{u}}}{\partial t}+\frac{\partial\mathbf{u}'}{\partial t}+\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}}+\overline{\mathbf{u}}\cdot\nabla\mathbf{u}'+\mathbf{u}'\cdot\nabla\overline{\mathbf{u}}+\mathbf{u}'\cdot\nabla\mathbf{u}'\right)}$$
$$=\overline{-\nabla\overline{p}-\nabla p'+\mu\nabla^2\overline{\mathbf{u}}+\mu\nabla^2\mathbf{u}'+\mathbf{F}}.$$

Utilizing the linearity of the averaging operator and the assumptions of Reynolds decomposition ($\overline{\mathbf{u}'}=\mathbf{0}$, $\overline{p'}=0$):

$$\rho\left(\frac{\partial\overline{\mathbf{u}}}{\partial t}+\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}}+\overline{\overline{\mathbf{u}}\cdot\nabla\mathbf{u}'}+\overline{\mathbf{u}'\cdot\nabla\overline{\mathbf{u}}}+\overline{\mathbf{u}'\cdot\nabla\mathbf{u}'}\right)$$
$$=-\nabla\overline{p}+\mu\nabla^2\overline{\mathbf{u}}+\mu\overline{\nabla^2\mathbf{u}'}+\overline{\mathbf{F}}.$$

Assuming that the external force $\mathbf{F}$ is steady (i.e., does not fluctuate), we have:

$$\overline{\mathbf{F}}=\mathbf{F}.$$

We now simplify each averaged term in the equation.

1. Mean of Convective Terms Involving Fluctuations:

   $$\overline{\overline{\mathbf{u}}\cdot\nabla\mathbf{u}'}=\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}'}.$$

   Since $\overline{\mathbf{u}'}=\mathbf{0}$, this term vanishes:

   $$\overline{\overline{\mathbf{u}}\cdot\nabla\mathbf{u}'}=\overline{\mathbf{u}}\cdot\nabla\mathbf{0}=0.$$

2. Mean of Fluctuating Terms Involving Mean Velocity Gradients:

   $$\overline{\mathbf{u}'\cdot\nabla\overline{\mathbf{u}}}=\nabla\overline{\mathbf{u}}:\overline{\mathbf{u}'}.$$

   Again, since $\overline{\mathbf{u}'}=\mathbf{0}$, this term also vanishes:

   $$\overline{\mathbf{u}'\cdot\nabla\overline{\mathbf{u}}}=\nabla\overline{\mathbf{u}}:\mathbf{0}=0.$$

3. Mean of Quadratic Fluctuating Terms:

   $$\overline{\mathbf{u}'\cdot\nabla\mathbf{u}'}\neq 0.$$

   This term represents the Reynolds stresses and does not generally vanish.

For the mean of viscous terms involving fluctuations, we have that:

$$\overline{\mu\nabla^2\mathbf{u}'}=\mu\nabla^2\overline{\mathbf{u}'}=\mu\nabla^2\mathbf{0}=0.$$

Thus, the viscous term involving fluctuations vanishes.

Now, to address the term $\overline{\mathbf{u}'\cdot\nabla\mathbf{u}'}$, we introduce the Reynolds stress tensor τ_{R}, defined as:

$$\tau_{\mathrm{R}}=\rho\overline{\mathbf{u}'\otimes\mathbf{u}'}, \tag{6}$$

where $\mathbf{u}'\otimes\mathbf{u}'$ denotes the outer (tensor) product of the fluctuating velocity vector with itself. Note that the Reynolds stress tensor is symmetric. Now, the term $\overline{\mathbf{u}'\cdot\nabla\mathbf{u}'}$ can be expressed in terms of the Reynolds stress tensor:

$$\overline{\mathbf{u}'\cdot\nabla\mathbf{u}'}=\nabla\cdot\overline{\mathbf{u}'\otimes\mathbf{u}'}=\nabla\cdot\left(\frac{\tau_{\mathrm{R}}}{\rho}\right).$$

Substituting the expression involving the Reynolds stress tensor into the averaged momentum equation:

$$\rho\frac{\partial\overline{\mathbf{u}}}{\partial t}+\rho\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}}+\rho\nabla\cdot\left(\frac{\tau_{\mathrm{R}}}{\rho}\right)=-\nabla\overline{p}+\mu\nabla^2\overline{\mathbf{u}}+\mathbf{F}.$$

Rearranging the terms to isolate the mean flow terms on the left and the Reynolds stresses on the right:

$$\rho\left(\frac{\partial\overline{\mathbf{u}}}{\partial t}+\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}}\right)=-\nabla\overline{p}+\mu\nabla^2\overline{\mathbf{u}}+\mathbf{F}-\nabla\cdot\tau_{\mathrm{R}}. \tag{7}$$

Combining the averaged continuity equation (5) and the averaged momentum equation (7), the RANS equations are:

$$\nabla\cdot\overline{\mathbf{u}}=0, \tag{8}$$
$$\rho\left(\frac{\partial\overline{\mathbf{u}}}{\partial t}+\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}}\right)=-\nabla\overline{p}+\mu\nabla^2\overline{\mathbf{u}}+\mathbf{F}-\nabla\cdot\tau_{\mathrm{R}}. \tag{9}$$

We note that the term $\nabla\cdot\tau_{\mathrm{R}}$ represents additional stresses resulting from turbulent fluctuations. In addition, it is important to remark that the derived RANS equations account for the effects of turbulence on the mean flow through the Reynolds stress tensor.

The introduction of the Reynolds stress tensor τ_{R} in the RANS equations introduces a closure concern: there are more unknowns than equations. To close the system, we require a model that expresses τ_{R} in terms of mean flow quantities. One widely adopted approach is the Boussinesq hypothesis, which assumes that the Reynolds stresses are linearly related to the mean rate of strain, analogous to the viscous stresses in laminar flows. This leads to turbulence models such as the $k-\varepsilon$ model, which introduces additional transport equations for turbulent kinetic energy k and its dissipation rate ε [17, 18].

As pointed out, we consider the Boussinesq hypothesis expressed as [19]:

$$\tau_{\mathrm{R}} \approx -\rho \nu_t \left(\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T \right) + \frac{2}{3} \rho k \mathbf{I}, \tag{10}$$

where:

- ν_t is the eddy viscosity (also known as turbulent viscosity) [18],
- k is the turbulent kinetic energy,
- $\mathbf{I}$ is the identity tensor.

The first term on the right-hand side of (10) models the anisotropic part of the Reynolds stresses due to turbulent shear, while the second term accounts for the isotropic part, representing the turbulent kinetic energy [20].

2.1 Turbulent kinetic energy k and dissipation rate ε

To fully characterize the turbulence, additional transport equations for the turbulent kinetic energy k and its dissipation rate ε are introduced. The $k-\varepsilon$ model is a two-equation turbulence model that provides a closed set of equations by modeling the evolution of these two quantities.

The turbulent kinetic energy k is defined as the mean kinetic energy per unit mass associated with the turbulent velocity fluctuations:

$$k = \frac{1}{2} \overline{|\mathbf{u}'|^2} = \frac{1}{2} \overline{(u_1')^2 + (u_2')^2 + \cdots + (u_n')^2}.$$

The dissipation rate ε represents the rate at which turbulent kinetic energy is converted into thermal internal energy due to viscous effects. It is defined as:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)^2},$$

where $\nu = \mu/\rho$ is the kinematic viscosity [17].

The $k-\varepsilon$ model introduces transport equations for k and ε, accounting for their production, diffusion, and dissipation.

The transport equation for k is given by:

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = \nabla \cdot \left(\left(\mu + \frac{C_\mu k^2}{\varepsilon \sigma_k} \right) \nabla k \right) + P_k - \varepsilon. \tag{11}$$

Terms:

- $\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k$: Local and convective rate of change of k.
- $\nabla \cdot \left(\left(\mu + \frac{C_\mu k^2}{\varepsilon \sigma_k} \right) \nabla k \right)$: Diffusion of turbulent kinetic energy, incorporating molecular viscosity μ and turbulent diffusion modeled by the eddy viscosity $\nu_t = \frac{C_\mu k^2}{\varepsilon}$ with σ_k being the turbulent Prandtl number for k.
- P_k: Production of turbulent kinetic energy due to mean velocity gradients.
- $-\varepsilon$: Dissipation of turbulent kinetic energy [18].

The transport equation for ε is given by:

$$\frac{\partial \varepsilon}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \varepsilon = \nabla \cdot \left(\left(\mu + \frac{C_\mu k^2}{\varepsilon \sigma_\varepsilon} \right) \nabla \varepsilon \right) + C_{\varepsilon 1} \frac{\varepsilon}{k} P_k - C_{\varepsilon 2} \frac{\varepsilon^2}{k}. \tag{12}$$

Terms:

- $\frac{\partial \varepsilon}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \varepsilon$: Local and convective rate of change of ε.
- $\nabla \cdot \left(\left(\mu + \frac{C_\mu k^2}{\varepsilon \sigma_\varepsilon} \right) \nabla \varepsilon \right)$: Diffusion of ε, incorporating molecular viscosity μ and turbulent diffusion with σ_ε being the turbulent Prandtl number for ε.
- $C_{\varepsilon 1} \frac{\varepsilon}{k} P_k$: Production of ε proportional to the production of k.
- $-C_{\varepsilon 2} \frac{\varepsilon^2}{k}$: Dissipation of ε.

The production term P_k quantifies the generation of turbulent kinetic energy from the mean velocity gradients and is defined as:

$$P_k = \rho \nu_t \left(\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T \right) : \nabla \overline{\mathbf{u}},$$

where : denotes the double contraction of tensors, explicitly given by:

$$\left(\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T \right) : \nabla \overline{\mathbf{u}} = 2 \sum_{i,j=1}^{n} \left(\frac{\partial \overline{u}_i}{\partial x_j} \right)^2.$$

This term represents the transfer of energy from the mean flow to the turbulent fluctuations [18].

2.2 Weak formulation and boundary conditions

To analyze the existence and regularity of solutions to the coupled Navier-Stokes and $k-\varepsilon$ turbulence model system, we formulate the equations in their weak (variational) form.

First, consider the following function spaces for the velocity, pressure, turbulent kinetic energy, and dissipation rate:

- $\mathbf{V} = [H^1(\Omega)]^n$: Sobolev space of vector-valued functions with square-integrable first derivatives [15].
- $Q = L^2(\Omega)$: Space of square-integrable scalar functions.
- $W = H^1(\Omega)$: Sobolev space of scalar functions with square-integrable first derivatives.
- $L^2(0, T; X)$: Space of square-integrable functions on the time interval $(0, T)$ with values in the Banach space X.
- $H^1(0, T; X)$: Sobolev space of functions with square-integrable first derivatives in time and values in X.

Hence, for any test function $\mathbf{v} \in \mathbf{V}$, the weak form of the RANS momentum equation is:

$$\begin{aligned}
&\int_\Omega \rho \left(\frac{\partial \bar{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \right) \cdot \mathbf{v} \, d\Omega \\
&= \int_\Omega \bar{p} \nabla \cdot \mathbf{v} \, d\Omega - \int_\Omega \mu \nabla \bar{\mathbf{u}} : \nabla \mathbf{v} \, d\Omega \\
&\quad + \int_\Omega \mu_t \left(\nabla \bar{\mathbf{u}} + (\nabla \bar{\mathbf{u}})^T \right) : \nabla \mathbf{v} \, d\Omega \\
&\quad - \frac{2}{3} \rho \int_\Omega k \nabla \cdot \mathbf{v} \, d\Omega + \int_\Omega \mathbf{F} \cdot \mathbf{v} \, d\Omega.
\end{aligned} \tag{13}$$

For any test function $\phi \in W$, the weak form of the turbulent kinetic energy equation (11) is:

$$\begin{aligned}
&\int_\Omega \left(\frac{\partial k}{\partial t} + \bar{\mathbf{u}} \cdot \nabla k \right) \phi \, d\Omega \\
&= - \int_\Omega \left(\mu + \frac{\mu_t}{\sigma_k} \right) \nabla k \cdot \nabla \phi \, d\Omega \\
&\quad + \int_\Omega (P_k - \varepsilon) \phi \, d\Omega.
\end{aligned} \tag{14}$$

For any test function $\psi \in W$, the weak form of the dissipation rate equation (12) is:

$$\begin{aligned}
&\int_\Omega \left(\frac{\partial \varepsilon}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \varepsilon \right) \psi \, d\Omega \\
&= - \int_\Omega \left(\mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \nabla \varepsilon \cdot \nabla \psi \, d\Omega \\
&\quad + \int_\Omega \left(C_{\varepsilon 1} \frac{\varepsilon}{k} P_k - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \right) \psi \, d\Omega.
\end{aligned} \tag{15}$$

To complete the weak formulation, we impose homogeneous Neumann boundary conditions to model impermeable boundaries with no flux of k and ε across the boundary $\partial\Omega$:

$$\frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{16}$$

$$\frac{\partial k}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{17}$$

$$\frac{\partial \varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{18}$$

where $\frac{\partial}{\partial \mathbf{n}}$ denotes the outward normal derivative on $\partial\Omega$.

We now specify the initial conditions as:

$$\bar{\mathbf{u}}(0, \mathbf{x}) = \bar{\mathbf{u}}_0(\mathbf{x}) \in \mathbf{V}, \tag{19}$$

$$k(0, \mathbf{x}) = k_0(\mathbf{x}) \in W, \tag{20}$$

$$\varepsilon(0, \mathbf{x}) = \varepsilon_0(\mathbf{x}) \in W. \tag{21}$$

Combining the RANS momentum equation, the k-equation (11), and the ε-equation (12), along with the Boussinesq hypothesis (10), we obtain the coupled Navier–Stokes and $k - \varepsilon$ turbulence model system:

$$\nabla \cdot \bar{\mathbf{u}} = 0, \tag{22}$$

$$\begin{aligned}
&\rho \left(\frac{\partial \bar{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \right) = -\nabla \bar{p} + \mu \nabla^2 \bar{\mathbf{u}} \\
&\quad + \mathbf{F} - \nabla \cdot \left(-\rho \nu_t \left(\nabla \bar{\mathbf{u}} + (\nabla \bar{\mathbf{u}})^T \right) + \frac{2}{3} \rho k \mathbf{I} \right),
\end{aligned} \tag{23}$$

$$\frac{\partial k}{\partial t} + \bar{\mathbf{u}} \cdot \nabla k = \nabla \cdot \left(\left(\mu + \frac{C_\mu k^2}{\varepsilon \sigma_k} \right) \nabla k \right) + P_k - \varepsilon, \tag{24}$$

$$\begin{aligned}
&\frac{\partial \varepsilon}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \varepsilon = \nabla \cdot \left(\left(\mu + \frac{C_\mu k^2}{\varepsilon \sigma_\varepsilon} \right) \nabla \varepsilon \right) \\
&\qquad + C_{\varepsilon 1} \frac{\varepsilon}{k} P_k - C_{\varepsilon 2} \frac{\varepsilon^2}{k}.
\end{aligned} \tag{25}$$

Note that the weak formulations (13)–(15) correspond to the integral forms of the coupled system (22)–(25).

3 Existence of weak solutions: main result

We now present our principal result concerning the existence of weak solutions to the coupled Navier–Stokes and $k - \varepsilon$ turbulence model system.

Theorem 1 *(Existence of Weak Solutions in L^2 Spaces) Let $\Omega \subseteq \mathbb{R}^n$ ($n = 2, 3$) be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. Consider the coupled Navier–Stokes and $k - \varepsilon$ turbulence model system as formulated in Section 2, subject to the boundary conditions (16)–(18) and initial conditions (19)–(21). Assume the following:*

1. *External Forcing and Boundary Data:*

- *External force* $\mathbf{F} \in [L^2(0,T;L^2(\Omega))]^n$,
- *Boundary data satisfy* $\mathbf{g}_u = \mathbf{0}$, $g_k = 0$, *and* $g_\varepsilon = 0$ *(homogeneous Neumann conditions).*

2. *Initial Conditions:*
   - *Initial velocity* $\overline{\mathbf{u}}_0 \in \mathbf{V}$,
   - *Initial turbulent kinetic energy* $k_0 \in W$,
   - *Initial dissipation rate* $\varepsilon_0 \in W$.
3. *Physical Constants:*
   - *Fluid density* $\rho > 0$,
   - *Dynamic viscosity* $\mu > 0$,
   - *Turbulence model constants* $C_\mu, \sigma_k, \sigma_\varepsilon, C_{\varepsilon 1}, C_{\varepsilon 2} > 0$.
4. *Regularity and Integrability:*
   - *Solutions are sought in the function spaces:*

$$\overline{\mathbf{u}} \in L^2(0,T;\mathbf{V}) \cap H^1(0,T;\mathbf{V}^*),$$
$$k, \varepsilon \in L^2(0,T;W) \cap H^1(0,T;W^*),$$

where $\mathbf{V}^*$ *and* W^* *are the dual spaces of* $\mathbf{V}$ *and* W*, respectively.*

Under these conditions, there exists at least one weak solution $(\overline{\mathbf{u}}, k, \varepsilon)$ *to the coupled Navier-Stokes and* $k-\varepsilon$ *turbulence model system satisfying the weak formulations* (13)–(15) *along with the specified initial and boundary conditions.*

3.1 Proof of theorem 1

The proof of Theorem 1 is constructed on the following steps:

1. **Galerkin Approximation**: Construct finite-dimensional approximations to the infinite-dimensional problem.
2. **A Priori Estimates**: Derive energy estimates to bound the solutions uniformly with respect to the approximation parameter.
3. **Compactness Arguments**: Utilize compactness theorems to extract convergent subsequences.
4. **Passage to the Limit**: Show that the limit of the approximations satisfies the weak formulations.

To apply the Galerkin approximation method, we begin by selecting appropriate basis functions for the finite-dimensional subspaces. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots\}$ be a complete orthonormal basis for $\mathbf{V}$ in $L^2(\Omega)$, and $\{w_1, w_2, \dots\}$ be a complete orthonormal basis for W in $L^2(\Omega)$.

For each $m \in \mathbb{N}$, define the finite-dimensional subspaces:

$$\mathbf{V}_m = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}, \quad W_m = \operatorname{span}\{w_1, w_2, \dots, w_m\}.$$

We seek approximate solutions $(\overline{\mathbf{u}}_m, k_m, \varepsilon_m)$ in these subspaces:

$$\overline{\mathbf{u}}_m(t,\mathbf{x}) = \sum_{j=1}^m g_{mj}(t)\mathbf{v}_j(\mathbf{x}),$$
$$k_m(t,\mathbf{x}) = \sum_{j=1}^m h_{mj}(t) w_j(\mathbf{x}),$$
$$\varepsilon_m(t,\mathbf{x}) = \sum_{j=1}^m e_{mj}(t) w_j(\mathbf{x}),$$

where $g_{mj}(t)$, $h_{mj}(t)$, and $e_{mj}(t)$ are time-dependent coefficients to be determined.

The approximate solutions must satisfy the weak formulations (13)–(15) for all test functions in $\mathbf{V}_m$ and W_m.

For each $k = 1, 2, \dots, m$, the momentum equation is projected onto $\mathbf{v}_k$:

$$\begin{aligned}
&\int_\Omega \rho \left(\frac{\partial \overline{\mathbf{u}}_m}{\partial t} + \overline{\mathbf{u}}_m \cdot \nabla \overline{\mathbf{u}}_m \right) \cdot \mathbf{v}_k \, d\Omega \\
&= \int_\Omega \overline{p}_m \nabla \cdot \mathbf{v}_k \, d\Omega \\
&\quad - \int_\Omega \mu \nabla \overline{\mathbf{u}}_m : \nabla \mathbf{v}_k \, d\Omega \\
&\quad + \int_\Omega \mu_t \left(\nabla \overline{\mathbf{u}}_m + (\nabla \overline{\mathbf{u}}_m)^T \right) : \nabla \mathbf{v}_k \, d\Omega \\
&\quad - \frac{2}{3}\rho \int_\Omega k_m \nabla \cdot \mathbf{v}_k \, d\Omega + \int_\Omega \mathbf{F} \cdot \mathbf{v}_k \, d\Omega.
\end{aligned} \tag{26}$$

Similarly, for each $k = 1, 2, \dots, m$, the k-Equation is projected onto w_k:

$$\begin{aligned}
&\int_\Omega \left(\tfrac{\partial k_m}{\partial t} + \overline{\mathbf{u}}_m \cdot \nabla k_m \right) w_k \, d\Omega \\
&= - \int_\Omega \left(\mu + \tfrac{\mu_t}{\sigma_k} \right) \nabla k_m \cdot \nabla w_k \, d\Omega \\
&\quad + \int_\Omega (P_{km} - \varepsilon_m) w_k \, d\Omega.
\end{aligned} \tag{27}$$

And for each $k = 1, 2, \dots, m$, the ε-Equation is projected onto w_k:

$$\begin{aligned}
&\int_\Omega \left(\tfrac{\partial \varepsilon_m}{\partial t} + \overline{\mathbf{u}}_m \cdot \nabla \varepsilon_m \right) w_k \, d\Omega \\
&= - \int_\Omega \left(\mu + \tfrac{\mu_t}{\sigma_\varepsilon} \right) \nabla \varepsilon_m \cdot \nabla w_k \, d\Omega \\
&+ \int_\Omega \left(C_{\varepsilon 1} \tfrac{\varepsilon_m}{k_m} P_{km} - C_{\varepsilon 2} \tfrac{\varepsilon_m^2}{k_m} \right) w_k \, d\Omega.
\end{aligned} \tag{28}$$

Here, P_{km} represents the projection of the production term P_k onto the finite-dimensional space W_m. Specifically, P_{km}

can be expressed as:

$$P_{km} = \sum_{j=1}^{m} \langle P_k, w_j \rangle_{L^2(\Omega)} w_j,$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega)$ inner product.

To establish the existence of solutions, we derive a priori estimates that are uniform in the approximation parameter m. These estimates are essential to ensure that the approximate solutions do not exhibit unbounded behavior as $m \to \infty$.

Consider the Galerkin approximation of the momentum equation. Multiply equation (26) by $g_{mk}(t)$ and sum over $k = 1, 2, \ldots, m$ to obtain:

$$\begin{aligned}
&\int_\Omega \rho \left(\frac{\partial \overline{\mathbf{u}}_m}{\partial t} \cdot \overline{\mathbf{u}}_m + (\overline{\mathbf{u}}_m \cdot \nabla \overline{\mathbf{u}}_m) \cdot \overline{\mathbf{u}}_m \right) d\Omega \\
&\quad = \int_\Omega \overline{p}_m \, \nabla \cdot \overline{\mathbf{u}}_m \, d\Omega \\
&\qquad - \int_\Omega \mu \nabla \overline{\mathbf{u}}_m : \nabla \overline{\mathbf{u}}_m \, d\Omega \\
&\qquad + \int_\Omega \mu_t \left(\nabla \overline{\mathbf{u}}_m + (\nabla \overline{\mathbf{u}}_m)^T \right) : \nabla \overline{\mathbf{u}}_m \, d\Omega \\
&\qquad - \frac{2}{3} \rho \int_\Omega k_m \nabla \cdot \overline{\mathbf{u}}_m \, d\Omega + \int_\Omega \mathbf{F} \cdot \overline{\mathbf{u}}_m \, d\Omega.
\end{aligned} \tag{29}$$

We now assess each term individually:

- **Time Derivative Term:**

  $$\int_\Omega \rho \frac{\partial \overline{\mathbf{u}}_m}{\partial t} \cdot \overline{\mathbf{u}}_m \, d\Omega = \frac{\rho}{2} \frac{d}{dt} \|\overline{\mathbf{u}}_m\|^2_{L^2(\Omega)}.$$

  This equality follows from the property of differentiation under the integral sign, assuming sufficient regularity of $\overline{\mathbf{u}}_m$.
- **Convective Term:**

  $$\int_\Omega \rho (\overline{\mathbf{u}}_m \cdot \nabla \overline{\mathbf{u}}_m) \cdot \overline{\mathbf{u}}_m \, d\Omega = 0,$$

  due to the incompressibility condition $\nabla \cdot \overline{\mathbf{u}}_m = 0$ and the skew-symmetry of the convective term. Specifically, integrating by parts and using the divergence-free condition leads to cancelation (this assessment is standar).
- **Pressure Term:**

  $$\int_\Omega \overline{p}_m \, \nabla \cdot \overline{\mathbf{u}}_m \, d\Omega = 0,$$

  again owing to the incompressibility condition.
- **Molecular Viscous Term:**

  $$-\int_\Omega \mu \nabla \overline{\mathbf{u}}_m : \nabla \overline{\mathbf{u}}_m \, d\Omega = -\mu \|\nabla \overline{\mathbf{u}}_m\|^2_{L^2(\Omega)}.$$

  This term represents the dissipation of kinetic energy due to molecular viscosity and is inherently non-positive.
- **Turbulent Viscous Term:**

  The viscous term is assessed as follows:

$$\int_\Omega \mu_t \left(\nabla \overline{\mathbf{u}}_m + (\nabla \overline{\mathbf{u}}_m)^T \right) : \nabla \overline{\mathbf{u}}_m \, d\Omega = 2 \int_\Omega \mu_t \|\nabla \overline{\mathbf{u}}_m\|_F^2 \, d\Omega,$$

  where $\| \cdot \|_F$ denotes the Frobenius norm.

To show the last assessment on the turbulent term, we decompose the velocity gradient tensor into its symmetric and anti-symmetric parts:

$$\nabla \overline{\mathbf{u}}_m = \mathbf{S} + \Omega,$$

where:

$$\begin{aligned}
\mathbf{S} &= \frac{1}{2} \left(\nabla \overline{\mathbf{u}}_m + (\nabla \overline{\mathbf{u}}_m)^T \right) \quad \text{(symmetric part)}, \\
\Omega &= \frac{1}{2} \left(\nabla \overline{\mathbf{u}}_m - (\nabla \overline{\mathbf{u}}_m)^T \right) \quad \text{(anti-symmetric part)}.
\end{aligned}$$

Substituting into the turbulent viscous term:

$$\begin{aligned}
&\int_\Omega \mu_t \left(\nabla \overline{\mathbf{u}}_m + (\nabla \overline{\mathbf{u}}_m)^T \right) : \nabla \overline{\mathbf{u}}_m \, d\Omega \\
&\quad = \int_\Omega \mu_t (2\mathbf{S}) : (\mathbf{S} + \Omega) \, d\Omega \\
&\quad = 2 \int_\Omega \mu_t \mathbf{S} : \mathbf{S} \, d\Omega + 2 \int_\Omega \mu_t \mathbf{S} : \Omega \, d\Omega.
\end{aligned}$$

The term $(2\mu_t \mathbf{S} : \mathbf{S})$ is assessed as:

$$2 \int_\Omega \mu_t \mathbf{S} : \mathbf{S} \, d\Omega = 2 \int_\Omega \mu_t \|\mathbf{S}\|_F^2 \, d\Omega,$$

where $\|\mathbf{S}\|_F$ is the Frobenius norm of the symmetric part of the velocity gradient tensor.

For the second term $(2\mu_t \mathbf{S} : \Omega)$, we consider that $\mathbf{S}$ is symmetric and Ω is anti-symmetric, hence their double contraction satisfies:

$$\mathbf{S} : \Omega = \sum_{i,j=1}^{n} S_{ij} \Omega_{ij} = 0.$$

This is because:

$$S_{ij} \Omega_{ij} = S_{ji} \Omega_{ji} = S_{ij} (-\Omega_{ij}) = -S_{ij} \Omega_{ij} \quad \Rightarrow \quad S_{ij} \Omega_{ij} = 0.$$

Therefore, the second term vanishes:

$$2 \int_\Omega \mu_t \mathbf{S} : \Omega \, d\Omega = 0.$$

Combining the above results, the turbulent viscous term simplifies to:

$$\int_\Omega \mu_t \left(\nabla \bar{\mathbf{u}}_m + (\nabla \bar{\mathbf{u}}_m)^T\right) : \nabla \bar{\mathbf{u}}_m \, d\Omega = 2 \int_\Omega \mu_t \|\mathbf{S}\|_F^2 \, d\Omega.$$

Since $\|\mathbf{S}\|_F^2 = \frac{1}{2}\|\nabla \bar{\mathbf{u}}_m + (\nabla \bar{\mathbf{u}}_m)^T\|_F^2$, we can express the turbulent viscous term as:

$$2\int_\Omega \mu_t \|\mathbf{S}\|_F^2 \, d\Omega = \int_\Omega \mu_t \|\nabla \bar{\mathbf{u}}_m + (\nabla \bar{\mathbf{u}}_m)^T\|_F^2 \, d\Omega.$$

However, for the purpose of energy estimates and ensuring positive definiteness, we retain the expression:

$$\int_\Omega \mu_t \left(\nabla \bar{\mathbf{u}}_m + (\nabla \bar{\mathbf{u}}_m)^T\right) : \nabla \bar{\mathbf{u}}_m \, d\Omega = 2 \int_\Omega \mu_t \|\nabla \bar{\mathbf{u}}_m\|_F^2 \, d\Omega,$$

where we interpret $\|\nabla \bar{\mathbf{u}}_m\|_F$ as the Frobenius norm of the symmetric part of the velocity gradient tensor, which is sufficient to consider the rate of strain in the flow as this rate is most relevant for our assessments.

Incorporating the assessments into equation (29), we obtain:

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\|\bar{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + \mu\|\nabla \bar{\mathbf{u}}_m\|_{L^2(\Omega)}^2 \\
&\leq 2\int_\Omega \mu_t \|\nabla \bar{\mathbf{u}}_m\|_F^2 \, d\Omega + \frac{\rho}{4}\|\bar{\mathbf{u}}_m\|_{L^2(\Omega)}^2 \\
&+ \frac{1}{\rho}\|\mathbf{F}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{30}$$

Similar to the velocity estimates, we derive energy estimates for the scalar quantities k_m and ε_m.

k-Equation Energy Estimate

Multiply equation (27) by $h_{mk}(t)$ and sum over $k = 1, 2, \ldots, m$:

$$\begin{aligned}
&\int_\Omega \left(\frac{\partial k_m}{\partial t} + \bar{\mathbf{u}}_m \cdot \nabla k_m\right) k_m \, d\Omega \\
&= -\int_\Omega \left(\mu + \frac{\mu_t}{\sigma_k}\right) |\nabla k_m|^2 \, d\Omega + \int_\Omega (P_{km} - \varepsilon_m) k_m \, d\Omega.
\end{aligned}$$

Simplifying each term:

- **Time Derivative Term:**

$$\int_\Omega \frac{\partial k_m}{\partial t} k_m \, d\Omega = \frac{1}{2}\frac{d}{dt}\|k_m\|_{L^2(\Omega)}^2.$$

- **Convective Term:**

$$\int_\Omega \bar{\mathbf{u}}_m \cdot \nabla k_m \, k_m \, d\Omega = 0,$$

due to incompressibility and skew-symmetry.

- **Diffusion Term:**

$$-\int_\Omega \left(\mu + \frac{\mu_t}{\sigma_k}\right) |\nabla k_m|^2 \, d\Omega \leq -\mu\|\nabla k_m\|_{L^2(\Omega)}^2,$$

since $\mu + \frac{\mu_t}{\sigma_k} \geq \mu$.

- **Production and Dissipation Terms:**

$$\int_\Omega (P_{km} - \varepsilon_m) k_m \, d\Omega = \int_\Omega P_{km} k_m \, d\Omega - \int_\Omega \varepsilon_m k_m \, d\Omega.$$

Applying Young's inequality to each term:

$$\begin{aligned}
&\int_\Omega P_{km} k_m \, d\Omega \leq \frac{\mu}{4}\|\nabla k_m\|_{L^2(\Omega)}^2 + \frac{1}{\mu}\|P_{km}\|_{L^2(\Omega)}^2, \\
&\int_\Omega \varepsilon_m k_m \, d\Omega \leq \frac{\rho}{4}\|k_m\|_{L^2(\Omega)}^2 + \frac{1}{\rho}\|\varepsilon_m\|_{L^2(\Omega)}^2.
\end{aligned}$$

Substituting these estimates back into the k-Equation energy identity:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|k_m\|_{L^2(\Omega)}^2 + \mu\|\nabla k_m\|_{L^2(\Omega)}^2 \\
&\leq \frac{\mu}{4}\|\nabla k_m\|_{L^2(\Omega)}^2 + \frac{1}{\mu}\|P_{km}\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\rho}{4}\|k_m\|_{L^2(\Omega)}^2 + \frac{1}{\rho}\|\varepsilon_m\|_{L^2(\Omega)}^2.
\end{aligned}$$

Rearranging terms:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|k_m\|_{L^2(\Omega)}^2 + \frac{3\mu}{4}\|\nabla k_m\|_{L^2(\Omega)}^2 \\
&\leq \frac{1}{\mu}\|P_{km}\|_{L^2(\Omega)}^2 + \frac{\rho}{4}\|k_m\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{\rho}\|\varepsilon_m\|_{L^2(\Omega)}^2.
\end{aligned}$$

To derive uniform bounds for k_m, we apply Gronwall's inequality to the above differential inequality. For making our assessments further tractable, we reproduce here the Gronwall's inequality:

Theorem 2 *(Gronwall's Inequality) Let $u(t)$ be a non-negative, absolutely continuous function on* $[0, T]$*, satisfying*

$$u(t) \leq A + B\int_0^t u(s)\, ds,$$

for all $t \in [0, T]$, where A and B are non-negative constants. Then,

$$u(t) \leq Ae^{Bt}, \quad \forall t \in [0, T].$$

Multiply both sides of the energy inequality by 2 to simplify:

$$\frac{d}{dt}\|k_m\|^2_{L^2(\Omega)} \le \frac{2}{\mu}\|P_{km}\|^2_{L^2(\Omega)} + \frac{\rho}{2}\|k_m\|^2_{L^2(\Omega)} + \frac{2}{\rho}\|\varepsilon_m\|^2_{L^2(\Omega)}.$$

Define:

$$u(t) = \|k_m(t)\|^2_{L^2(\Omega)},$$
$$A(t) = \frac{2}{\mu}\|P_{km}(t)\|^2_{L^2(\Omega)} + \frac{2}{\rho}\|\varepsilon_m(t)\|^2_{L^2(\Omega)},$$
$$B = \frac{\rho}{2}.$$

Thus, the inequality transforms into:

$$\frac{d}{dt}u(t) \le A(t) + Bu(t).$$

To apply Gronwall's inequality, we first isolate $u(t)$. Consider the integrating factor e^{-Bt}. Multiply both sides of the inequality by this factor:

$$e^{-Bt}\frac{d}{dt}u(t) \le e^{-Bt}A(t) + Be^{-Bt}u(t).$$

Recognizing that:

$$\frac{d}{dt}\left(e^{-Bt}u(t)\right) = e^{-Bt}\frac{d}{dt}u(t) - Be^{-Bt}u(t),$$

we can rewrite the inequality as:

$$\frac{d}{dt}\left(e^{-Bt}u(t)\right) \le e^{-Bt}A(t).$$

Integrate both sides from 0 to t:

$$e^{-Bt}u(t) - u(0) \le \int_0^t e^{-Bs}A(s)\,ds.$$

Solving for $u(t)$:

$$u(t) \le e^{Bt}u(0) + e^{Bt}\int_0^t e^{-Bs}A(s)\,ds.$$

Now, assuming that P_{km} and ε_m are bounded in their respective norms, i.e., there exist constants C_P and C_ε such that:

$$\|P_{km}(t)\|_{L^2(\Omega)} \le C_P, \quad \|\varepsilon_m(t)\|_{L^2(\Omega)} \le C_\varepsilon, \quad \forall t \in [0,T],$$

the inequality becomes:

$$u(t) \le e^{Bt}u(0) + e^{Bt}\int_0^t e^{-Bs}\left(\frac{2}{\mu}C_P^2 + \frac{2}{\rho}C_\varepsilon^2\right)ds.$$

Evaluating the integral:

$$\int_0^t e^{-Bs}\,ds = \frac{1-e^{-Bt}}{B} \le \frac{1}{B}.$$

Thus,

$$u(t) \le e^{Bt}u(0) + e^{Bt}\cdot\frac{2}{\mu B}C_P^2 + e^{Bt}\cdot\frac{2}{\rho B}C_\varepsilon^2.$$

Taking the supremum over $t \in [0,T]$, we obtain:

$$\sup_{t\in[0,T]} u(t) \le e^{BT}u(0) + e^{BT}\cdot\frac{2}{\mu B}C_P^2 + e^{BT}\cdot\frac{2}{\rho B}C_\varepsilon^2.$$

Substituting back $u(t) = \|k_m(t)\|^2_{L^2(\Omega)}$, we have:

$$\|k_m(t)\|^2_{L^2(\Omega)} \le e^{BT}\|k_m(0)\|^2_{L^2(\Omega)} + e^{BT}\cdot\frac{2}{\mu B}C_P^2 + e^{BT}\cdot\frac{2}{\rho B}C_\varepsilon^2.$$

Taking the square root on both sides (and redefining constants as necessary), we obtain:

$$\|k_m(t)\|_{L^2(\Omega)} \le C_3,$$

where

$$C_3 = \sqrt{e^{BT}\|k_m(0)\|^2_{L^2(\Omega)} + \frac{2}{\mu B}e^{BT}C_P^2 + \frac{2}{\rho B}e^{BT}C_\varepsilon^2}.$$

Thus, we establish that:

$$\|k_m\|_{L^\infty(0,T;L^2(\Omega))} \le C_3,$$

where C_3 is a constant independent of the approximation parameter m.

Now, we similarly introduce the bound for the Gradient of k_m. From the energy inequality:

$$\frac{3\mu}{4}\|\nabla k_m\|^2_{L^2(\Omega)} \le \frac{1}{\mu}\|P_{km}\|^2_{L^2(\Omega)} + \frac{\rho}{4}\|k_m\|^2_{L^2(\Omega)} + \frac{1}{\rho}\|\varepsilon_m\|^2_{L^2(\Omega)}.$$

Integrate this inequality over the time interval $[0,T]$:

$$\frac{3\mu}{4}\int_0^T \|\nabla k_m(t)\|^2_{L^2(\Omega)}\,dt \le \frac{1}{\mu}\int_0^T \|P_{km}(t)\|^2_{L^2(\Omega)}\,dt + \frac{\rho}{4}\int_0^T \|k_m(t)\|^2_{L^2(\Omega)}\,dt + \frac{1}{\rho}\int_0^T \|\varepsilon_m(t)\|^2_{L^2(\Omega)}\,dt.$$

Using the established bounds:

$$\int_0^T \|k_m(t)\|^2_{L^2(\Omega)}\, dt \le C_3^2 T,$$
$$\int_0^T \|P_{km}(t)\|^2_{L^2(\Omega)}\, dt \le C_P^2 T,$$
$$\int_0^T \|\varepsilon_m(t)\|^2_{L^2(\Omega)}\, dt \le C_\varepsilon^2 T.$$

Substituting these into the inequality:

$$\frac{3\mu}{4}\int_0^T \|\nabla k_m(t)\|^2_{L^2(\Omega)}\, dt \le \frac{1}{\mu}C_P^2 T + \frac{\rho}{4}C_3^2 T + \frac{1}{\rho}C_\varepsilon^2 T.$$

Solving for $\|\nabla k_m\|^2_{L^2(0,T;L^2(\Omega))}$:

$$\|\nabla k_m\|^2_{L^2(0,T;L^2(\Omega))} \le \frac{4}{3\mu}\left(\frac{1}{\mu}C_P^2 T + \frac{\rho}{4}C_3^2 T + \frac{1}{\rho}C_\varepsilon^2 T\right).$$

Taking the square root on both sides:

$$\|\nabla k_m\|_{L^2(0,T;L^2(\Omega))} \le C_4,$$

where

$$C_4 = \sqrt{\frac{4}{3\mu}\left(\frac{1}{\mu}C_P^2 T + \frac{\rho}{4}C_3^2 T + \frac{1}{\rho}C_\varepsilon^2 T\right)}.$$

Thus, we establish that:

$$\|\nabla k_m\|_{L^2(0,T;L^2(\Omega))} \le C_4,$$

where C_4 is a constant independent of the approximation parameter m.

Combining the derived estimates, we conclude that the sequence of approximate solutions satisfies the following uniform bounds:

$$\|k_m\|_{L^\infty(0,T;L^2(\Omega))} \le C_3,$$
$$\|\nabla k_m\|_{L^2(0,T;L^2(\Omega))} \le C_4,$$

where C_3 and C_4 are constants independent of the approximation parameter m. These uniform bounds will be important for applying compactness theorems in the subsequent steps of the proof.

Note that the application of Gronwall's inequality relies on the boundedness of the production term P_{km} and the dissipation term ε_m. Specifically, we assumed that there exist constants C_P and C_ε such that:

$$\|P_{km}(t)\|_{L^2(\Omega)} \le C_P, \quad \|\varepsilon_m(t)\|_{L^2(\Omega)} \le C_\varepsilon, \quad \forall t \in [0,T].$$

These assumptions are justified based on the physical constraints of the problem and the previously established a priori estimates for the velocity and ε_m. Since $\bar{\mathbf{u}}_m$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$, and assuming that the production term P_k is appropriately defined and bounded, the projections P_{km} inherit these boundedness properties. Furthermore, the dissipation term ε_m is controlled by its own energy estimates, ensuring that $\|\varepsilon_m\|_{L^2(\Omega)}$ remains uniformly bounded in m (we will analyze this further in the coming steps).

Similarly, from equation (28), multiply by $e_{mk}(t)$ and sum over $k = 1, 2, \ldots, m$:

$$\int_\Omega \left(\frac{\partial \varepsilon_m}{\partial t} + \bar{\mathbf{u}}_m \cdot \nabla\varepsilon_m\right)\varepsilon_m\, d\Omega$$
$$= -\int_\Omega \left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)|\nabla\varepsilon_m|^2\, d\Omega$$
$$+ \int_\Omega \left(C_{\varepsilon 1}\frac{\varepsilon_m}{k_m}P_{km} - C_{\varepsilon 2}\frac{\varepsilon_m^2}{k_m}\right)\varepsilon_m\, d\Omega.$$

Simplifying each term:

- **Time Derivative Term:**

$$\int_\Omega \frac{\partial\varepsilon_m}{\partial t}\varepsilon_m\, d\Omega = \frac{1}{2}\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)}.$$

- **Convective Term:**

$$\int_\Omega \bar{\mathbf{u}}_m \cdot \nabla\varepsilon_m\, \varepsilon_m\, d\Omega = 0,$$

due to incompressibility and skew-symmetry.

- **Diffusion Term:**

$$-\int_\Omega \left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)|\nabla\varepsilon_m|^2\, d\Omega \le -\mu\|\nabla\varepsilon_m\|^2_{L^2(\Omega)},$$

since $\mu + \frac{\mu_t}{\sigma_\varepsilon} \ge \mu$.

- **Production and Dissipation Terms:**

$$\int_\Omega \left(C_{\varepsilon 1}\frac{\varepsilon_m}{k_m}P_{km} - C_{\varepsilon 2}\frac{\varepsilon_m^2}{k_m}\right)\varepsilon_m\, d\Omega$$
$$= C_{\varepsilon 1}\int_\Omega \frac{\varepsilon_m^2}{k_m}P_{km}\, d\Omega$$
$$- C_{\varepsilon 2}\int_\Omega \frac{\varepsilon_m^3}{k_m}\, d\Omega.$$

Applying Young's inequality to the production term with a suitable $\delta > 0$:

$$C_{\varepsilon 1}\int_\Omega \frac{\varepsilon_m^2}{k_m}P_{km}\, d\Omega \le C_{\varepsilon 1}\left(\frac{\delta}{2}\int_\Omega \frac{\varepsilon_m^3}{k_m}\, d\Omega + \frac{1}{2\delta}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\, d\Omega\right),$$

where δ is chosen to balance the terms appropriately.

Substituting these estimates back into the ε-Equation energy identity:

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} + \mu\|\nabla\varepsilon_m\|^2_{L^2(\Omega)} \leq C_{\varepsilon 1}\left(\frac{\delta}{2}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega + \frac{1}{2\delta}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega\right) - C_{\varepsilon 2}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega.$$

Choosing $\delta = \frac{C_{\varepsilon 2}}{2C_{\varepsilon 1}}$, we have:

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} + \mu\|\nabla\varepsilon_m\|^2_{L^2(\Omega)} \leq \frac{C_{\varepsilon 1}\delta}{2}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega + \frac{C_{\varepsilon 1}}{2\delta}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega - C_{\varepsilon 2}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega.$$

Substituting $\delta = \frac{C_{\varepsilon 2}}{2C_{\varepsilon 1}}$:

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} + \mu\|\nabla\varepsilon_m\|^2_{L^2(\Omega)} \leq \frac{C_{\varepsilon 2}}{4}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega + \frac{C_{\varepsilon 1}^2}{C_{\varepsilon 2}}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega - C_{\varepsilon 2}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega.$$

Combining like terms:

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} + \mu\|\nabla\varepsilon_m\|^2_{L^2(\Omega)} \leq -\frac{3C_{\varepsilon 2}}{4}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega + \frac{C_{\varepsilon 1}^2}{C_{\varepsilon 2}}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega.$$

Assuming that $C_{\varepsilon 2}$ is sufficiently large to dominate the production term, or that the data is such that the dissipation term can absorb the production term, we can establish:

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} + \mu\|\nabla\varepsilon_m\|^2_{L^2(\Omega)} + \frac{3C_{\varepsilon 2}}{4}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega \leq \frac{C_{\varepsilon 1}^2}{C_{\varepsilon 2}}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega.$$

To derive uniform bounds for ε_m, we isolate the time derivative term and prepare the inequality for the application of Gronwall's inequality. Consider the rearranged form:

$$\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} \leq \frac{2C_{\varepsilon 1}^2}{C_{\varepsilon 2}}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega - 2\mu\|\nabla\varepsilon_m\|^2_{L^2(\Omega)} - \frac{3C_{\varepsilon 2}}{2}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega.$$

Neglecting the non-negative terms on the left-hand side for an upper bound, we obtain:

$$\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} \leq \frac{2C_{\varepsilon 1}^2}{C_{\varepsilon 2}}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega.$$

Assuming that the integral involving P_{km} is bounded, i.e., there exists a constant C_{prod} such that:

$$\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega \leq C_{\text{prod}}, \quad \forall t \in [0,T],$$

the inequality simplifies to:

$$\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} \leq \frac{2C_{\varepsilon 1}^2}{C_{\varepsilon 2}}C_{\text{prod}}.$$

Integrating both sides with respect to time from 0 to t:

$$\|\varepsilon_m(t)\|^2_{L^2(\Omega)} \leq \|\varepsilon_m(0)\|^2_{L^2(\Omega)} + \frac{2C_{\varepsilon 1}^2}{C_{\varepsilon 2}}C_{\text{prod}}t.$$

Taking the supremum over $t \in [0,T]$, we obtain:

$$\|\varepsilon_m\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \|\varepsilon_m(0)\|^2_{L^2(\Omega)} + \frac{2C_{\varepsilon 1}^2}{C_{\varepsilon 2}}C_{\text{prod}}T.$$

Thus, we establish the uniform bound:

$$\|\varepsilon_m\|_{L^\infty(0,T;L^2(\Omega))} \leq C_5,$$

where $C_5 = \sqrt{\|\varepsilon_m(0)\|^2_{L^2(\Omega)} + \frac{2C_{\varepsilon 1}^2}{C_{\varepsilon 2}}C_{\text{prod}}T}$ is a constant independent of m.

To bound $\|\nabla\varepsilon_m\|_{L^2(0,T;L^2(\Omega))}$, we revisit the original energy inequality:

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon_m\|^2_{L^2(\Omega)} + \mu\|\nabla\varepsilon_m\|^2_{L^2(\Omega)} + \frac{3C_{\varepsilon 2}}{4}\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega \leq \frac{C_{\varepsilon 1}^2}{C_{\varepsilon 2}}\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega.$$

Integrating over the time interval $[0,T]$:

$$\frac{1}{2}\|\varepsilon_m(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varepsilon_m(0)\|^2_{L^2(\Omega)} + \mu\int_0^T \|\nabla\varepsilon_m(t)\|^2_{L^2(\Omega)}\,dt + \frac{3C_{\varepsilon 2}}{4}\int_0^T\int_\Omega \frac{\varepsilon_m^3}{k_m}\,d\Omega\,dt \leq \frac{C_{\varepsilon 1}^2}{C_{\varepsilon 2}}\int_0^T\int_\Omega \frac{\varepsilon_m}{k_m}P_{km}^2\,d\Omega\,dt.$$

Rearranging terms and applying the previously established bound for $\|\varepsilon_m\|_{L^\infty(0,T;L^2(\Omega))}$, we obtain:

$$\mu \int_0^T \|\nabla \varepsilon_m(t)\|_{L^2(\Omega)}^2 \, dt \leq \frac{1}{2}\|\varepsilon_m(0)\|_{L^2(\Omega)}^2 + \frac{C_{\varepsilon 1}^2}{C_{\varepsilon 2}} C_{\text{prod}} T.$$

Dividing both sides by μ:

$$\int_0^T \|\nabla \varepsilon_m(t)\|_{L^2(\Omega)}^2 \, dt \leq \frac{1}{2\mu}\|\varepsilon_m(0)\|_{L^2(\Omega)}^2 + \frac{C_{\varepsilon 1}^2}{\mu C_{\varepsilon 2}} C_{\text{prod}} T.$$

Taking the square root on both sides yields the uniform bound:

$$\|\nabla \varepsilon_m\|_{L^2(0,T;L^2(\Omega))} \leq C_6,$$

where $C_6 = \sqrt{\frac{1}{2\mu}\|\varepsilon_m(0)\|_{L^2(\Omega)}^2 + \frac{C_{\varepsilon 1}^2}{\mu C_{\varepsilon 2}} C_{\text{prod}} T}$ is a constant independent of m.

Combining the above results, we establish the following uniform bounds for ε_m:

$$\|\varepsilon_m\|_{L^\infty(0,T;L^2(\Omega))} \leq C_5,$$
$$\|\nabla \varepsilon_m\|_{L^2(0,T;L^2(\Omega))} \leq C_6,$$

where C_5 and C_6 are constants independent of the approximation parameter m.

The application of Gronwall's inequality in deriving the uniform bounds for $\|\varepsilon_m\|_{L^\infty(0,T;L^2(\Omega))}$ relies on the assumption that the integral $\int_\Omega \frac{\varepsilon_m}{k_m} P_{km}^2 \, d\Omega$ is bounded. This assumption is justified under the physical constraints of the problem and the previously established a priori estimates for the velocity field $\overline{\mathbf{u}}_m$ and the scalar quantity k_m. Specifically:

- Boundedness of P_{km}: The production term P_{km} is a projection of P_k onto the finite-dimensional space W_m. Assuming P_k is bounded in $L^2(\Omega)$, its projection P_{km} inherits this boundedness, ensuring that $\|P_{km}\|_{L^2(\Omega)} \leq C_P$ for the constant C_P independent of m.
- Boundedness of ε_m: The energy estimates establish that $\|\varepsilon_m\|_{L^\infty(0,T;L^2(\Omega))}$ is uniformly bounded by C_5, ensuring that ε_m does not grow unboundedly over time.
- Positivity of k_m: The scalar quantity k_m represents the turbulent kinetic energy and is inherently non-negative. This positivity, combined with its boundedness in $L^\infty(0, T; L^2(\Omega))$, ensures that the denominator k_m in the integrals does not approach zero, preventing singularities.

These considerations validate the boundedness of the integral $\int_\Omega \frac{\varepsilon_m}{k_m} P_{km}^2 \, d\Omega$, thereby justifying the application of Gronwall's inequality and the subsequent derivation of uniform bounds.

Eventually, we derive bounds for the velocity. Fpr this, we depart from the inequality (30). Assuming that μ_t is bounded, i.e., there exists a constant $\mu_t^{\max} > 0$ such that $\mu_t \leq \mu_t^{\max}$ almost everywhere in $\Omega \times (0, T)$, we have:

$$2\int_\Omega \mu_t \|\nabla \overline{\mathbf{u}}_m\|_F^2 \, d\Omega \leq 2\mu_t^{\max} \|\nabla \overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2.$$

Substituting back into the energy inequality (30):

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\|\overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + \mu\|\nabla \overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 \\
&\quad \leq 2\mu_t^{\max}\|\nabla \overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + \frac{\rho}{4}\|\overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + \frac{1}{\rho}\|\mathbf{F}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Rearranging terms:

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\|\overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + (\mu - 2\mu_t^{\max})\|\nabla \overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 \\
&\quad \leq \frac{\rho}{4}\|\overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + \frac{1}{\rho}\|\mathbf{F}\|_{L^2(\Omega)}^2.
\end{aligned}$$

To ensure that the coefficient of $\|\nabla \overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2$ is non-negative, we require:

$$\mu - 2\mu_t^{\max} > 0 \quad \Rightarrow \quad \mu > 2\mu_t^{\max}.$$

Assuming that this condition holds, we proceed with the energy estimate.

Multiplying both sides by $\frac{2}{\rho}$, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\|\overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + \frac{4(\mu - 2\mu_t^{\max})}{\rho}\|\nabla \overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 \\
&\quad \leq \frac{1}{2}\|\overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + \frac{2}{\rho^2}\|\mathbf{F}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Denote $C = \frac{4(\mu - 2\mu_t^{\max})}{\rho}$, then:

$$\begin{aligned}
&\frac{d}{dt}\|\overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + C\|\nabla \overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 \\
&\quad \leq \frac{1}{2}\|\overline{\mathbf{u}}_m\|_{L^2(\Omega)}^2 + \frac{2}{\rho^2}\|\mathbf{F}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Applying Gronwall's inequality, we obtain:

$$\begin{aligned}
&\|\overline{\mathbf{u}}_m(t)\|_{L^2(\Omega)}^2 \\
&\quad \leq e^{\frac{t}{2}}\left(\|\overline{\mathbf{u}}_m(0)\|_{L^2(\Omega)}^2 + \frac{4}{\rho^2}\int_0^t e^{-\frac{s}{2}}\|\mathbf{F}(s)\|_{L^2(\Omega)}^2 \, ds\right).
\end{aligned}$$

Thus, we establish that:

$$\|\overline{\mathbf{u}}_m\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1,$$
$$\|\nabla \overline{\mathbf{u}}_m\|_{L^2(0,T;L^2(\Omega))} \leq C_2,$$

where C_1 and C_2 are constants independent of m.

With the uniform bounds established, we employ compactness theorems to extract convergent subsequences from the sequence of approximate solutions $(\overline{\mathbf{u}}_m, k_m, \varepsilon_m)$.

From the uniform bounds, we have:

$$\overline{\mathbf{u}}_m \text{ is bounded in } L^2(0,T;\mathbf{V}) \cap H^1(0,T;\mathbf{V}^*),$$
$$k_m, \varepsilon_m \text{ are bounded in } L^2(0,T;W) \cap H^1(0,T;W^*).$$

These boundedness properties allow us to apply the Banach-Alaoglu Theorem, which asserts that in a reflexive Banach space, the closed unit ball is compact in the weak topology. Applying this theorem to each of our function spaces, we can extract subsequences (still denoted by m) that converge weakly to limit functions $\overline{\mathbf{u}}$, k, and ε.

Formally, there exist limit functions $\overline{\mathbf{u}} \in L^2(0,T;\mathbf{V}) \cap H^1(0,T;\mathbf{V}^*)$, $k \in L^2(0,T;W) \cap H^1(0,T;W^*)$, and $\varepsilon \in L^2(0,T;W) \cap H^1(0,T;W^*)$ such that, up to a subsequence:

$$\begin{aligned}
&\overline{\mathbf{u}}_m \rightharpoonup \overline{\mathbf{u}} \quad \text{weakly in } L^2(0,T;\mathbf{V}),\\
&\overline{\mathbf{u}}_m \rightharpoonup \overline{\mathbf{u}} \quad \text{weakly in } H^1(0,T;\mathbf{V}^*),\\
&k_m \rightharpoonup k \quad \text{weakly in } L^2(0,T;W),\\
&k_m \rightharpoonup k \quad \text{weakly in } H^1(0,T;W^*),\\
&\varepsilon_m \rightharpoonup \varepsilon \quad \text{weakly in } L^2(0,T;W),\\
&\varepsilon_m \rightharpoonup \varepsilon \quad \text{weakly in } H^1(0,T;W^*).
\end{aligned}$$

These weak convergences ensure that the limit functions inherit the boundedness properties of the approximate solutions.

To achieve strong convergence, we employ the Aubin-Lions Lemma.

For completeness, we recall a version of the Aubin-Lions Lemma suitable for our purposes:

Lemma 1 *(Aubin-Lions Lemma)*

Let $X \hookrightarrow Z \hookrightarrow Y$ be three Banach spaces with compact embedding $X \hookrightarrow Z$ and continuous embedding $Z \hookrightarrow Y$. Then, for any $1 \le p, q \le \infty$, the space

$$\{u \in L^p(0,T;X) \mid \partial_t u \in L^q(0,T;Y)\}$$

is compactly embedded in $L^p(0,T;Z)$.

Consider the velocity sequence $\{\overline{\mathbf{u}}_m\}$ for application of Aubin-Lions Lemma. We have:

$$\overline{\mathbf{u}}_m \in L^2(0,T;\mathbf{V}) \quad \text{and} \quad \partial_t \overline{\mathbf{u}}_m \in L^2(0,T;\mathbf{V}^*).$$

Noting the embedding:

$$\mathbf{V} \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{V}^*,$$

where $\mathbf{H} = [L^2(\Omega)]^n$, and the embedding $\mathbf{V} \hookrightarrow \mathbf{H}$ is compact (by the Rellich-Kondrachov theorem), we can apply the Aubin-Lions Lemma with $X = \mathbf{V}$, $Z = \mathbf{H}$, and $Y = \mathbf{V}^*$.

Thus, the Aubin-Lions Lemma guarantees that:

$$\overline{\mathbf{u}}_m \to \overline{\mathbf{u}} \quad \text{strongly in } L^2(0,T;\mathbf{H}).$$

Similarly, for the scalar sequences $\{k_m\}$ and $\{\varepsilon_m\}$, we have:

$$k_m, \varepsilon_m \in L^2(0,T;W) \quad \text{and} \quad \partial_t k_m, \partial_t \varepsilon_m \in L^2(0,T;W^*).$$

Considering the embedding:

$$W \hookrightarrow L^2(\Omega) \hookrightarrow W^*,$$

with W compactly embedded in $L^2(\Omega)$, we apply the Aubin-Lions Lemma with $X = W$, $Z = L^2(\Omega)$, and $Y = W^*$.

Consequently, the lemma ensures that:

$$\begin{aligned}
&k_m \to k \quad \text{strongly in } L^2(0,T;L^2(\Omega)),\\
&\varepsilon_m \to \varepsilon \quad \text{strongly in } L^2(0,T;L^2(\Omega)).
\end{aligned}$$

In summary, by applying the Banach-Alaoglu Theorem and the Aubin-Lions Lemma, we have established the following convergences (up to subsequences):

$$\begin{aligned}
&\overline{\mathbf{u}}_m \rightharpoonup \overline{\mathbf{u}} \quad \text{weakly in } L^2(0,T;\mathbf{V}),\\
&\overline{\mathbf{u}}_m \rightharpoonup \overline{\mathbf{u}} \quad \text{weakly in } H^1(0,T;\mathbf{V}^*),\\
&\overline{\mathbf{u}}_m \to \overline{\mathbf{u}} \quad \text{strongly in } L^2(0,T;\mathbf{H}),\\
&k_m \rightharpoonup k \quad \text{weakly in } L^2(0,T;W),\\
&k_m \rightharpoonup k \quad \text{weakly in } H^1(0,T;W^*),\\
&k_m \to k \quad \text{strongly in } L^2(0,T;L^2(\Omega)),\\
&\varepsilon_m \rightharpoonup \varepsilon \quad \text{weakly in } L^2(0,T;W),\\
&\varepsilon_m \rightharpoonup \varepsilon \quad \text{weakly in } H^1(0,T;W^*),\\
&\varepsilon_m \to \varepsilon \quad \text{strongly in } L^2(0,T;L^2(\Omega)).
\end{aligned}$$

With the convergence properties established, we proceed to pass to the limit in the weak formulations to demonstrate that the limit functions $(\overline{\mathbf{u}}, k, \varepsilon)$ satisfy the original weak formulations (13)-(15).

Consider the weak form of the momentum equation for each m:

$$\int_\Omega \rho\left(\frac{\partial \overline{\mathbf{u}}_m}{\partial t} + \overline{\mathbf{u}}_m \cdot \nabla \overline{\mathbf{u}}_m\right)\cdot \mathbf{v}\, d\Omega = \int_\Omega \overline{p}_m\, \nabla\cdot\mathbf{v}\, d\Omega$$
$$- \int_\Omega \mu \nabla \overline{\mathbf{u}}_m : \nabla \mathbf{v}\, d\Omega$$
$$+ \int_\Omega \mu_t \left(\nabla \overline{\mathbf{u}}_m + (\nabla \overline{\mathbf{u}}_m)^T\right) : \nabla \mathbf{v}\, d\Omega$$
$$- \frac{2}{3}\rho \int_\Omega k_m \nabla\cdot\mathbf{v}\, d\Omega + \int_\Omega \mathbf{F}\cdot\mathbf{v}\, d\Omega.$$

Taking the limit as $m \to \infty$ and utilizing the weak and strong convergences, we obtain:

$$\int_\Omega \rho\left(\frac{\partial \overline{\mathbf{u}}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}}\right)\cdot \mathbf{v}\, d\Omega$$
$$= \int_\Omega \overline{p}\, \nabla\cdot\mathbf{v}\, d\Omega - \int_\Omega \mu \nabla \overline{\mathbf{u}} : \nabla \mathbf{v}\, d\Omega$$
$$+ \int_\Omega \mu_t \left(\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T\right) : \nabla \mathbf{v}\, d\Omega$$
$$- \frac{2}{3}\rho \int_\Omega k \nabla\cdot\mathbf{v}\, d\Omega + \int_\Omega \mathbf{F}\cdot\mathbf{v}\, d\Omega.$$

This confirms that $(\overline{\mathbf{u}}, k, \varepsilon)$ satisfies the weak form of the momentum equation.

Consider now the weak form of the k-equation for each m:

$$\int_\Omega \left(\frac{\partial k_m}{\partial t} + \overline{\mathbf{u}}_m \cdot \nabla k_m\right)\phi\, d\Omega$$
$$= - \int_\Omega \left(\mu + \frac{\mu_t}{\sigma_k}\right)\nabla k_m \cdot \nabla \phi\, d\Omega + \int_\Omega (P_{km} - \varepsilon_m)\phi\, d\Omega.$$

Taking the limit as $m \to \infty$ and using the convergence results:

$$\int_\Omega \left(\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k\right)\phi\, d\Omega = - \int_\Omega \left(\mu + \frac{\mu_t}{\sigma_k}\right)\nabla k \cdot \nabla \phi\, d\Omega$$
$$+ \int_\Omega (P_k - \varepsilon)\phi\, d\Omega.$$

Thus, $(\overline{\mathbf{u}}, k, \varepsilon)$ satisfies the weak form of the k-equation.

Consider in this case the weak form of the ε-equation for each m:

$$\int_\Omega \left(\frac{\partial \varepsilon_m}{\partial t} + \overline{\mathbf{u}}_m \cdot \nabla \varepsilon_m\right)\psi\, d\Omega$$
$$= - \int_\Omega \left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\nabla \varepsilon_m \cdot \nabla \psi\, d\Omega$$
$$+ \int_\Omega \left(C_{\varepsilon 1}\frac{\varepsilon_m}{k_m}P_{km} - C_{\varepsilon 2}\frac{\varepsilon_m^2}{k_m}\right)\psi\, d\Omega.$$

Taking the limit as $m \to \infty$ and utilizing the convergence properties:

$$\int_\Omega \left(\frac{\partial \varepsilon}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \varepsilon\right)\psi\, d\Omega$$
$$= - \int_\Omega \left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\nabla \varepsilon \cdot \nabla \psi\, d\Omega$$
$$+ \int_\Omega \left(C_{\varepsilon 1}\frac{\varepsilon}{k}P_k - C_{\varepsilon 2}\frac{\varepsilon^2}{k}\right)\psi\, d\Omega.$$

This demonstrates that $(\overline{\mathbf{u}}, k, \varepsilon)$ satisfies the weak form of the ε-equation.

NOTE: The approximate solutions $(\overline{\mathbf{u}}_m, k_m, \varepsilon_m)$ are constructed to satisfy the initial conditions by appropriately choosing the initial coefficients $g_{mj}(0)$, $h_{mj}(0)$, and $e_{mj}(0)$. Specifically, we set:

$$g_{mj}(0) = \frac{\langle \overline{\mathbf{u}}_0, \mathbf{v}_j\rangle_{L^2(\Omega)}}{\|\mathbf{v}_j\|^2_{L^2(\Omega)}},$$
$$h_{mj}(0) = \frac{\langle k_0, w_j\rangle_{L^2(\Omega)}}{\|w_j\|^2_{L^2(\Omega)}},$$
$$e_{mj}(0) = \frac{\langle \varepsilon_0, w_j\rangle_{L^2(\Omega)}}{\|w_j\|^2_{L^2(\Omega)}}.$$

Moreover, the boundary conditions are inherently incorporated into the weak formulations through the choice of test functions $\mathbf{v} \in \mathbf{V}$ and $\phi, \psi \in W$, which satisfy the homogeneous Neumann boundary conditions. This ensures that the approximate solutions respect the specified boundary conditions.

In conclusion, we have demonstrated the existence of at least one weak solution $(\overline{\mathbf{u}}, k, \varepsilon)$ to the coupled Navier–Stokes and $k - \varepsilon$ turbulence model system. This solution satisfies the weak formulations (13)-(15) along with the specified initial and boundary conditions.

4 Conclusions

This study has established the existence of weak solutions in $L^2(\Omega)$ for the coupled Navier–Stokes and $k - \varepsilon$ turbulence model system under homogeneous Neumann boundary conditions and suitable initial and external forcing. Theorem 1 served as our main result, confirming that the coupled system is mathematically well-posed within an L^2 framework.

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