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Self-Organized Criticality and Energy Cascades: A Proposal for a Toy Model to Approach Fluid Turbulence

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Abstract

Self-organized criticality (SOC) describes a class of dynamical systems that may evolve toward statistically critical states characterized by scale-free avalanche-like events. In this work, we study an SOC-inspired discrete toy model and examine the avalanche-size statistics generated by local stochastic interactions. The aim is to explore whether a minimal avalanche model can reproduce statistical features that are formally reminiscent of multiscale turbulent phenomenology. We present a mathematical formulation of the toy model, analyze its numerical avalanche-size distribution, and discuss its relation to concepts of scaling, intermittency, and energy cascades in turbulence. The comparison with Navier–Stokes turbulence is therefore interpreted as a qualitative and statistical analogy, not as a physically complete correspondence. The results suggest that SOC-inspired toy models can provide a useful exploratory framework for understanding heavy-tailed activity and multiscale organization.

Keywords: energy cascade; self-organized criticality (SOC); turbulence; power-law distribution

MSC: 76F55; 76D05; 60K35; 82C41

1. Introduction

Self-organized criticality (SOC), introduced by Bak, Tang, and Wiesenfeld in 1987 [1], represents a paradigm for understanding complex systems that naturally evolve toward a critical state. SOC systems are characterized by avalanche-like events that lack a characteristic scale, leading to power-law distributions in event sizes and durations. In this framework, small, local disturbances propagate across the system, triggering avalanches of various magnitudes. This concept has broad applicability across fields, including geophysics, neuroscience, ecology, and economics [2–4]. Notably, SOC has also motivated analogies in fluid mechanics, where turbulent flows exhibit multiscale energy cascades, intermittent bursts, and scaling behavior that can be formally compared with avalanche dynamics in SOC systems [5,6].

The central hypothesis in SOC is that systems with local interactions, such as the rearrangement of tiles in a cellular automaton, may self-organize into a critical state without the need for external parameter tuning. These systems exhibit scale invariance, where small perturbations can lead to events of very different magnitudes. The power-law distribution of avalanche sizes, described by the probability $P(S) \sim S^{-\tau}$ for large S, is a hallmark of SOC. The exponent τ typically depends on the underlying system dynamics and defines the scaling properties of the avalanches [7].

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The possible connection between SOC and turbulence must be formulated with care. In fluid mechanics, the energy cascade process transfers kinetic energy from large scales, where it is injected, toward smaller scales, where viscous effects become dominant. This transfer across scales is only partially analogous to the propagation of avalanches in SOC models. Kolmogorov's theory of turbulence [8] provides a statistical description of inertial-range scaling, while later refinements and experimental studies have shown that real turbulent flows also display intermittency and deviations from simple self-similar scaling [9–11]. Therefore, the purpose of the present work is not to claim that SOC provides a complete physical model of turbulence but to explore whether a simplified SOC-inspired toy model can reproduce statistical signatures that are formally reminiscent of multiscale turbulent phenomenology.

The specific gap addressed in this paper concerns the role of simplified discrete models as conceptual tools for comparing avalanche-like dynamics with turbulent cascade phenomenology. The SOC literature has extensively studied power laws, avalanches, and critical states in lattice systems [2,7], while the turbulence literature has developed a precise framework for energy spectra, structure functions, intermittency, and viscous dissipation [5,6,10]. What remains less direct is how far a minimal stochastic lattice model can be used to build a controlled analogy with turbulence without overstating its physical validity. This paper addresses this issue by explicitly distinguishing between qualitative analogy, statistical similarity, and physically grounded modeling.

1.1. Toy Model for SOC

We begin by introducing a discrete toy model that captures some basic features of SOC. Let $\Omega \subset \mathbb{Z}^2$ represent a two-dimensional lattice of sites, and let $T : \Omega \times \mathbb{N} \to \{0, 1, \ldots, N\}$ denote the state of the system at discrete time $t \in \mathbb{N}$. Here, $T(\mathbf{x}, t)$ represents the tile type at site $\mathbf{x} \in \Omega$. The system evolves according to local stochastic rules in which neighboring tiles satisfying a prescribed matching condition are removed, producing a cascade. When a tile is removed, nearby tiles may collapse into the empty space, triggering further removals in a cascading fashion.

The dynamics of the system are governed by a stochastic Markov process, where the transition operator $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ maps the system state space $\mathcal{S}$ at time t to the state at time $t + 1$. The transition probabilities depend on the local configuration of tiles. Let S represent the size of an avalanche, defined as the number of tiles removed during a cascade, and let $P(S)$ denote the probability of an avalanche of size S. The expected SOC-like behavior is expressed as

$$P(S) \sim S^{-\tau}, \quad \text{as } S \to \infty, \tag{1}$$

where τ is the power-law exponent that characterizes the scaling behavior of the system. In classical SOC models, such scaling can be related to singular behavior of suitable generating functions or finite-size scaling relations in the critical regime [7]. In the present toy model, however, Equation (1) is treated as a numerical scaling hypothesis to be tested, not as an exact analytical result derived from the update rule.

In this toy model, the avalanches represent cascading events in which local interactions may lead to system-wide consequences. The simplicity of the model makes it useful for examining scale-free statistics, but its fundamental properties should not be assumed to extend directly to continuous turbulent flows. Rather, the model is used here as an exploratory framework for comparing certain statistical signatures of SOC with selected features of turbulent cascade phenomenology.

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1.2. Connection to Fluid Mechanics and Turbulence

Turbulence is typically described by the incompressible Navier–Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}(\mathbf{x}, t) \in H^1(\Omega)$ is the velocity field, $p(\mathbf{x}, t) \in L^2(\Omega)$ is the pressure, $\nu > 0$ is the kinematic viscosity, and $\mathbf{f} \in L^2(\Omega)$ represents an external forcing. The domain $\Omega \subset \mathbb{R}^3$ is a bounded region, typically with periodic or no-slip boundary conditions.

In turbulent flows, energy is injected at large scales through the external forcing term $\mathbf{f}$. The nonlinear convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is responsible for inertial transport and nonlinear interactions among scales. The term $\nu \nabla^2 \mathbf{u}$ represents the diffusion of momentum due to viscous stresses. Energy dissipation is obtained from the associated viscous contribution in the kinetic-energy balance, rather than from interpreting $\nu \nabla^2 \mathbf{u}$ itself as an energy dissipation term. For sufficiently regular incompressible flows, a common expression for the viscous dissipation rate is

$$\varepsilon = \nu \int_{\Omega} |\nabla \mathbf{u}|^2 \, d\mathbf{x}, \tag{3}$$

up to normalization by the measure of the domain when a spatial average is considered.

The statistical properties of turbulence are often described through velocity increments, energy spectra, structure functions, and local or coarse-grained dissipation fields. These quantities may display scaling behavior and intermittency, especially at high Reynolds numbers [5,6,8–10]. In this context, comparisons with SOC should be understood as qualitative and statistical. Similar to avalanche sizes in SOC systems, some turbulent observables may exhibit broad distributions and scale-dependent fluctuations. However, the energy dissipation rate ε is a dimensional physical quantity derived from the velocity gradient, whereas the avalanche size S in the toy model is a dimensionless event count. Consequently, any comparison between S and ε requires caution and cannot by itself establish a physical equivalence between the two systems.

Experimental and numerical studies of turbulence [11] confirm the presence of scale-dependent intermittency, suggesting that turbulence may exhibit some SOC-like statistical features. The energy cascade in turbulence is, therefore, considered here only as an analogy to avalanche dynamics in the toy model. This analogy may help organize certain aspects of multiscale behavior, but it does not replace the Navier–Stokes description of fluid motion and does not imply that both systems belong to the same universality class.

1.3. Summary of the Paper

This paper is organized as follows:

- In Section 2, we present the mathematical formulation of the toy model, including the transition operator and the stochastic dynamics of the system. We discuss the avalanche-size distribution and the critical properties expected from SOC-like behavior.
- In Section 3, we review the fluid-dynamical quantities needed for the comparison, focusing on the energy cascade, viscous dissipation, structure functions, and intermittency in turbulent flows.
- Section 4 provides a numerical analysis of the toy model, including simulations of avalanche events and the resulting power-law fits. The interpretation of these numerical findings is restricted to the level of statistical analogy with turbulence-inspired quantities.
- Section 5 concludes the paper by summarizing the scope and limitations of the SOC–turbulence analogy.

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This paper does not establish a physical derivation of turbulent energy cascades from SOC dynamics. Instead, it examines whether a minimal stochastic avalanche model can reproduce scale-free and intermittent statistical features that are formally reminiscent of turbulence. The results are therefore intended as an exploratory contribution to conceptual modeling of multiscale systems, not as a predictive model of full-scale fluid flow.

2. Mathematical Formulation of the Toy Model

In this section, we present the mathematical formulation of the toy model for self-organized criticality (SOC). The model is defined on a finite two-dimensional lattice and is governed by local stochastic rules. Its purpose is not to reproduce the dynamics of a physical fluid but to generate avalanche-like events whose statistical properties can be analyzed and later compared, at a qualitative level, with some multiscale features of turbulent flows.

Let $\Omega \subset \mathbb{Z}^2$ be a finite two-dimensional lattice with $M \times N$ sites. To avoid a conflict between the number of lattice sites and the number of tile types, we denote by $q+1$ the number of possible tile states. Thus, each site $(i,j) \in \Omega$ is assigned a state

$$T(i,j,t) \in \{0,1,\dots,q\}$$

at a discrete time $t \in \mathbb{N}$. The value $T(i,j,t)$ represents the tile type at the site (i,j) at time t. The dynamics are determined by local matching rules: neighboring tiles satisfying a prescribed condition may be removed, and this removal may trigger further removals in adjacent or dynamically connected sites.

The configuration of the system at time t is represented by the state vector

$$\mathbf{T}(t) = (T(i,j,t))_{(i,j)\in\Omega}. \tag{4}$$

The finite state space of all possible configurations is

$$\mathcal{S} = \{0,1,\dots,q\}^{\Omega}. \tag{5}$$

At each time step, a local update is performed. This update may generate an avalanche, understood as the complete cascade of tile removals caused by a single initial triggering event.

The evolution of the system is described by a stochastic transition mechanism. More precisely, if $\mathbf{T}(t) = \mathbf{A} \in \mathcal{S}$, the next configuration is sampled according to transition probabilities

$$\mathbb{P}(\mathbf{T}(t+1) = \mathbf{B} \mid \mathbf{T}(t) = \mathbf{A}) = K(\mathbf{A},\mathbf{B}), \qquad \mathbf{A},\mathbf{B} \in \mathcal{S}, \tag{6}$$

where K is a Markov transition kernel satisfying

$$K(\mathbf{A},\mathbf{B}) \geq 0, \qquad \sum_{\mathbf{B}\in\mathcal{S}} K(\mathbf{A},\mathbf{B}) = 1.$$

Equivalently, one may write

$$\mathbf{T}(t+1) = \mathcal{T}(\mathbf{T}(t),\omega_t), \tag{7}$$

where $\mathcal{T}$ is a stochastic update map and ω_t denotes the random variables used in the update. This notation avoids interpreting $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ as a deterministic operator.

For each pair of neighboring sites (i,j) and (i',j'), the local rule assigns a probability $p_{(i,j),(i',j')}$ that an interaction is triggered when

$$T(i,j,t) = T(i',j',t).$$

If an interaction occurs, the corresponding tiles are removed, and the local configuration is updated. The resulting vacancies are filled according to the prescribed collapse rule, which may create new matching pairs and therefore continue the cascade. The detailed form of K depends on these local removal and collapse rules.

Let $P_t(\mathbf{A}) = \mathbb{P}(\mathbf{T}(t) = \mathbf{A})$ be the probability distribution of configurations at time t. Since the model evolves in discrete time, the distribution satisfies the Markov evolution equation

$$P_{t+1}(\mathbf{B}) = \sum_{\mathbf{A}\in\mathcal{S}} P_t(\mathbf{A})K(\mathbf{A},\mathbf{B}), \qquad \mathbf{B} \in \mathcal{S}. \tag{8}$$

This replaces the continuous-time master equation, which would require transition rates rather than one-step transition probabilities. A continuous-time formulation could also be introduced, but the numerical model considered in this paper is discrete.

In this model, an *avalanche* is defined as the finite sequence of tile removals generated by one triggering event before the system reaches a stable configuration again. The avalanche size, denoted by $S \in \mathbb{N}$, is the total number of tiles removed during this cascade. The main statistical object of interest is the avalanche-size distribution

$$P(S) = \mathbb{P}(\text{an avalanche has size } S).$$

For SOC systems, a central empirical signature is the possible emergence of a power-law regime,

$$P(S) \sim S^{-\tau}, \qquad S \to \infty, \tag{9}$$

where $\tau > 0$ is the avalanche-size exponent [7,12]. In the present work, Equation (9) is not assumed to be an exact theorem derived from the update rule. Rather, it is treated as a scaling hypothesis to be evaluated numerically. This distinction is important because the identification of power laws in finite numerical data requires care, and visual linearity in log–log plots is not by itself sufficient evidence of a genuine asymptotic power law [13,14].

A useful formal device for describing avalanche statistics is the generating function

$$G(z) = \sum_{S=0}^{\infty} P(S)z^S. \tag{10}$$

If the distribution has a power-law tail of the form (9), then the behavior of $G(z)$ near its radius of convergence reflects the large-S behavior of $P(S)$. In classical critical systems, non-analytic behavior of generating functions or finite-size scaling functions is associated with scale invariance. In the present toy model, however, this observation is used only as a formal motivation for the numerical study; a complete analytical derivation of the singular behavior of $G(z)$ is not provided.

In a finite lattice, the avalanche-size distribution cannot be a pure power law for all S. There is necessarily a finite-size cutoff determined by the lattice dimensions and the update rules. A more cautious finite-size scaling form is therefore

$$P_L(S) \simeq S^{-\tau} F\left(\frac{S}{S_c(L)}\right), \tag{11}$$

where L denotes a characteristic linear size of the lattice, $S_c(L)$ is a cutoff avalanche size, and F is a scaling function that decays sufficiently fast for large arguments. This form emphasizes that the exponent estimated from numerical simulations should be interpreted as an effective exponent over the observable scaling range, not as a universal constant.

The critical properties of the model are therefore assessed through the behavior of $P(S)$ in the large-S range available in the finite simulation. In some SOC models, such as sandpile models and forest-fire models, avalanche exponents depend on dimension, conservation properties, boundary conditions, driving mechanisms, and the precise update rules [7,12]. Consequently, the exponent τ obtained in the present toy model should not be described as universal in the strong sense. It is a model-dependent exponent or effective numerical exponent.

If the model is parametrized by an interaction probability p, one may also introduce a correlation length ξ associated with the typical spatial extent of avalanches. In idealized critical behavior, this length may diverge as

$$\xi \sim |p - p_c|^{-\nu_c}, \tag{12}$$

where p_c is a critical value, and ν_c is a correlation-length exponent. We use the notation ν_c to avoid confusion with the kinematic viscosity ν appearing later in the Navier–Stokes equations. For the finite stochastic model considered here, Equation (12) should be regarded as a standard critical-scaling template rather than as a proved property of the specific update rule.

The evolution of the system takes place in the discrete configuration space $\mathcal{S}$. Since $\mathcal{S}$ is finite, transition probabilities may be regarded as entries of a stochastic matrix, or equivalently as functions

$$K : \mathcal{S} \times \mathcal{S} \to [0, 1].$$

Thus, it is not necessary to introduce an $L^2(\mathcal{S})$ functional framework for the transition rates unless one passes to a continuum or infinite-state limit. In the present finite model, the Markov-chain formulation is sufficient and is better aligned with the numerical simulations reported below.

The mathematical formulation, therefore, supports the following limited claim: the model is capable of producing avalanche statistics that may be tested for SOC-like scaling. Whether these statistics are genuinely scale-free, and whether their exponents can be meaningfully compared with those from other systems, must be decided through numerical analysis with appropriate statistical caution.

3. Energy Cascade in Turbulent Flows

In this section, we recall the main fluid-dynamical concepts needed to compare the toy model with turbulence. The connection developed here is deliberately limited. Turbulence is a physical phenomenon governed by the conservation of mass and momentum, whereas the toy model introduced above is a discrete stochastic system. Therefore, the comparison should be understood as a qualitative and statistical analogy, not as a derivation of turbulent dynamics from SOC.

Turbulence is characterized by chaotic and multiscale behavior. In three-dimensional flows, kinetic energy is typically injected at large scales, transferred across intermediate scales by nonlinear interactions, and dissipated at small scales by viscosity. This process is usually referred to as the energy cascade. It is one of the central ideas in the statistical theory of turbulence [5,6,8]. The purpose of this section is to identify which aspects of this cascade can be compared with avalanche statistics in the toy model and which aspects remain specific to Navier–Stokes dynamics.

The dynamics of incompressible turbulent flows are governed by the Navier–Stokes equations, which express conservation of momentum together with incompressibility. For an incompressible fluid, these equations are written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \tag{13}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{14}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the velocity field, $p = p(\mathbf{x}, t)$ is the pressure divided by a constant density, $\nu > 0$ is the kinematic viscosity, and $\mathbf{f}$ is an external forcing term. The domain $\Omega \subset \mathbb{R}^3$ is assumed to be supplied with suitable boundary conditions, for example, periodic or no-slip boundary conditions.

For the purposes of the present discussion, one may assume sufficient regularity, for instance $\mathbf{u}(\cdot, t) \in H^1(\Omega)$ and $\mathbf{f}(\cdot, t) \in L^2(\Omega)$, so that the energy balance can be formally justified. This functional setting is used here only to make the notation meaningful; it is not intended as a full discussion of existence, uniqueness, or regularity for the Navier–Stokes equations.

The term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ represents nonlinear advection of momentum and is responsible for inertial interactions among scales. The term $\nu \nabla^2 \mathbf{u}$ represents diffusion of momentum due to viscous stresses. It should not be identified directly with energy dissipation. Rather, viscous energy dissipation is obtained from the contribution of the viscous term to the kinetic-energy balance. The incompressibility condition, Equation (14), expresses conservation of mass for a constant-density incompressible fluid.

Energy Cascade and Dissipation

The energy cascade describes the transfer of kinetic energy from large scales to smaller scales. Energy is injected into the flow through forcing or boundary motion at scales comparable to an integral length scale. Nonlinear interactions redistribute this energy across scales. At sufficiently small scales, viscous effects become dominant, and kinetic energy is converted into heat.

For sufficiently regular incompressible flows, the viscous dissipation rate may be written, up to normalization by the volume of the domain, as

$$\varepsilon = \nu \int_{\Omega} |\nabla \mathbf{u}|^2 \, d\mathbf{x}. \tag{15}$$

In homogeneous turbulence, one often uses the corresponding spatial or ensemble average. Equivalent formulations may involve the rate-of-strain tensor, depending on the normalization convention. The relevant point for the present paper is that ε is a physical quantity derived from velocity gradients and viscosity, whereas the avalanche size in the toy model is a dimensionless count of removed tiles.

In fully developed turbulence, a central phenomenological assumption is that the mean energy dissipation rate remains finite in the high-Reynolds-number limit. This is sometimes referred to as the dissipative anomaly and is part of the classical phenomenology of turbulence [10]. This statement concerns the mean dissipation rate and should not be confused with the assertion that all dissipation events follow a universal power-law distribution.

Kolmogorov's 1941 theory [8] provides a statistical description of the inertial range. In that range, energy injection and viscous dissipation are not dominant at the considered scales, and the energy spectrum is expected to follow

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}, \tag{16}$$

where k is the wavenumber and ε is the mean energy dissipation rate. More precisely, the proportionality constant is the Kolmogorov constant, and the relation is expected to hold only over an inertial range at a sufficiently high Reynolds number. This scaling law describes a statistical transfer of energy across scales, not a lossless transfer at every instant or every point of the flow.

Real turbulent flows also exhibit intermittency. Kolmogorov's refined similarity hypothesis and later multifractal approaches account for deviations from the simplest self-similar picture [6,9,11]. Experimental and numerical studies show that small-scale velocity gradients and coarse-grained dissipation fields may have broad, non-Gaussian distributions [10,11]. For this reason, it is more accurate to say that turbulence displays intermittent, scale-dependent fluctuations of dissipation, rather than to assume a universal power law of the form $P(\varepsilon) \sim \varepsilon^{-\beta}$ for all turbulent regimes.

This distinction is important for the analogy with SOC. In the toy model, the main observable is the avalanche size S, whose distribution may display a scaling range of the form

$$P(S) \sim S^{-\tau}, \qquad S \to \infty. \tag{17}$$

In turbulence, by contrast, the fundamental observables are velocity increments, energy spectra, structure functions, velocity gradients, and dissipation fields. Some of these quantities display scaling or heavy-tailed behavior, but they are not identical to avalanche sizes. Thus, Equation (17) should be compared with turbulent statistics only at the level of formal analogy.

The Reynolds number,

$$Re = \frac{UL}{\nu}, \tag{18}$$

where U is a characteristic velocity and L is a characteristic length scale, measures the relative importance of inertial and viscous effects. Large Reynolds numbers are associated with turbulent regimes and extended inertial ranges. However, the transition to turbulence at high Re should not be described as the attainment of a SOC critical state. Turbulence and SOC may share some statistical features, such as multiscale activity and intermittent events, but their governing mechanisms are different.

Intermittency in turbulence can be quantified through higher-order statistics. For example, the flatness of a velocity derivative may be written as

$$F = \frac{\left\langle (\partial u_1/\partial x_1)^4 \right\rangle}{\left\langle (\partial u_1/\partial x_1)^2 \right\rangle^2}. \tag{19}$$

For a Gaussian variable, $F = 3$. Values significantly larger than 3 indicate heavy tails and enhanced probability of extreme gradient events. In the toy model, an analogous diagnostic may be constructed from avalanche sizes or avalanche-size increments, but such a quantity should be interpreted as an intermittency-like indicator, not as the same physical observable used in fluid turbulence.

Another standard tool in turbulence is the family of longitudinal structure functions. If

$$\delta u_\ell(\mathbf{x}) = (\mathbf{u}(\mathbf{x} + \boldsymbol{\ell}) - \mathbf{u}(\mathbf{x})) \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|},$$

then the p-th order longitudinal structure function is

$$S_p(\ell) = \langle |\delta u_\ell(\mathbf{x})|^p \rangle, \qquad \ell = |\boldsymbol{\ell}|. \tag{20}$$

The second-order structure function scales as

$$S_2(\ell) \sim (\varepsilon \ell)^{2/3}. \tag{21}$$

Higher-order structure functions may deviate from the linear scaling prediction because of intermittency [6,9,11].

The toy model does not contain a velocity field, pressure, incompressibility constraint, viscosity, or nonlinear momentum equation. Therefore, its avalanche statistics cannot be used to derive Equations (16) and (21). What can be investigated is more modest:

whether the distribution of avalanche sizes and possible avalanche-based analogues of intermittency display broad scaling behavior reminiscent of some statistical features of turbulent cascades.

4. Numerical Analysis of the Toy Model

In this section, we present numerical simulations of the toy model described in Section 2. The goal is to examine whether the avalanche-size distribution generated by the model displays a broad scaling range compatible with SOC-like behavior. The numerical analysis is exploratory. Therefore, the results are interpreted as evidence of scale-free tendencies in the toy model, not as proof that the model belongs to a given universality class or as a quantitative model of fluid turbulence.

4.1. Simulation Setup

The toy model was implemented on a two-dimensional grid of size 100×100, with the system evolving over 10^5 discrete time steps. At each time step, a random tile from the grid was selected, and a cascade was triggered according to the local interaction rules described in Section 2. The avalanche size was recorded as the number of tiles removed during the corresponding cascade event.

The probability of triggering a local interaction was set to $p = 0.05$. This value was kept fixed throughout the simulations reported here. Consequently, the numerical results should be understood as corresponding to one representative parameter choice, rather than as a complete exploration of the model's phase space. A systematic analysis over different values of p, different grid sizes, and different random seeds would be required to establish robustness, finite-size scaling, and convergence properties.

To avoid excessive recursion depth issues, the recursive implementation of the cascading mechanism was replaced with an iterative stack-based approach. This modification does not change the local cascade rule; it only provides a more stable computational implementation for large cascades.

4.2. Results

The avalanche-size distribution was analyzed by plotting the empirical frequencies on a log–log scale. Figure 1 shows the empirical avalanche-size distribution together with a fitted power-law trend over the observed scaling range. The estimated exponent is approximately

$$\tau \approx 1.32.$$

This value should be regarded as an effective numerical exponent obtained from the finite simulation. It is not claimed to be universal, and it should not be directly identified with exponents associated with turbulent dissipation.

Figure 2 shows the corresponding histogram of avalanche sizes. The distribution is dominated by small avalanches, while larger avalanches occur less frequently. This behavior is consistent with the presence of a heavy-tailed distribution. However, following standard cautions in the statistical analysis of power laws, visual linearity in a log–log plot is not sufficient by itself to establish a genuine asymptotic power law [13,14]. More robust confirmation would require maximum-likelihood estimation, goodness-of-fit tests, comparison with alternative heavy-tailed distributions, and uncertainty estimates for the fitted exponent.

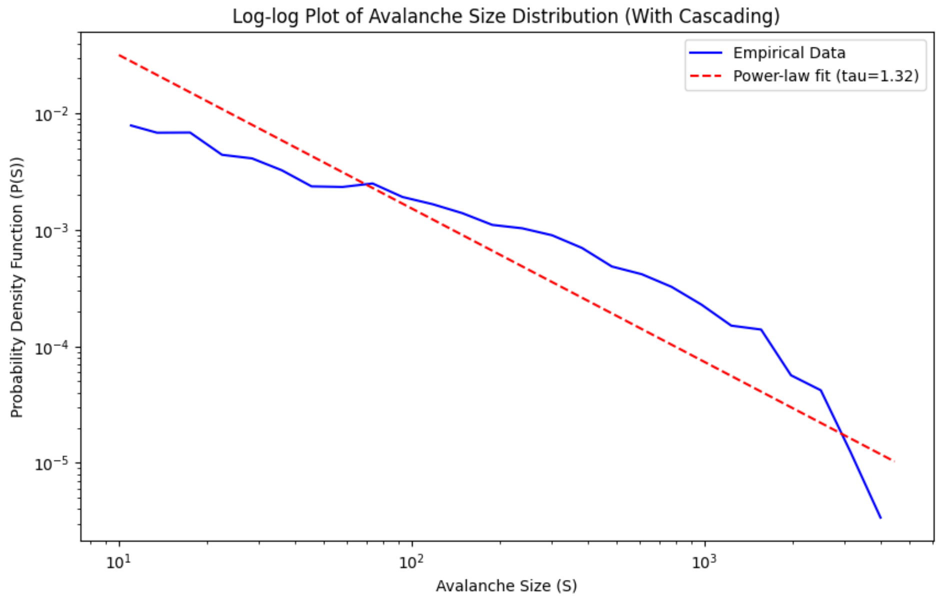

Figure 1. Log–log plot of the avalanche-size distribution with cascading. The empirical distribution is compared with a power-law fit over the observed scaling range. The fitted exponent is approximately $\tau \approx 1.32$. This estimate should be interpreted as an effective exponent for the finite simulation, not as a universal critical exponent.

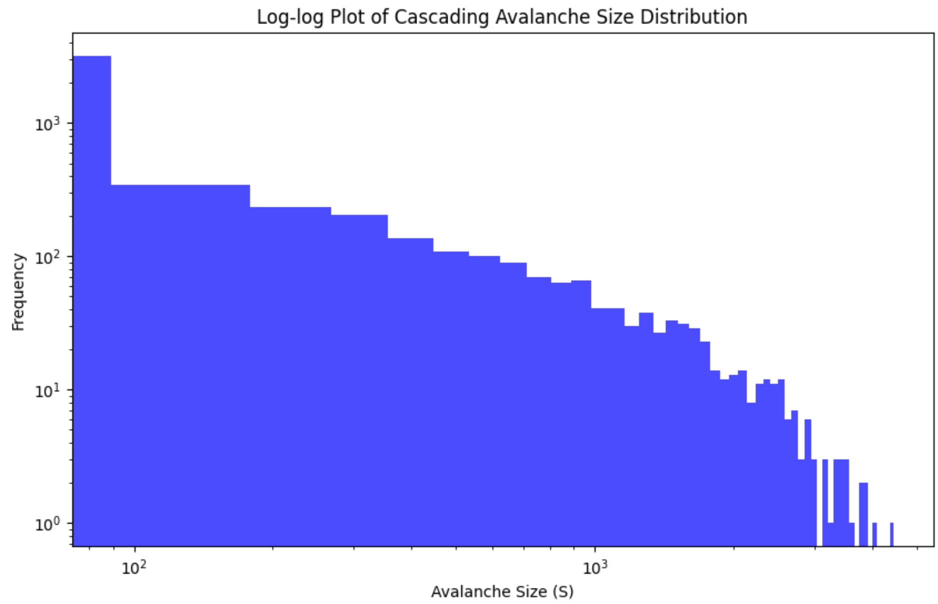

Figure 2. Histogram of cascading avalanche sizes. Small avalanches dominate the distribution, while larger avalanches appear with lower frequency. The figure supports the presence of a broad distribution, although additional statistical tests are required to confirm an asymptotic power-law regime.

4.3. Discussion

The numerical simulations show that the toy model generates avalanche events over a range of sizes. The empirical distribution presents a broad tail and can be fitted, over the observed range, by a power-law trend with an effective exponent $\tau \approx 1.32$. This result is compatible with SOC-like phenomenology in the limited sense that the model produces many small events and fewer large events. Nevertheless, the result should not be overstated. The exponent is obtained from a finite grid, a fixed triggering probability, and a single numerical protocol.

The histogram in Figure 2 supports the same qualitative interpretation: small avalanches dominate the dynamics, while larger events remain possible. This is reminiscent of avalanche statistics in SOC models such as sandpile models and forest-fire models [7,12]. However, different SOC models may have different exponents, and the value obtained here should be regarded as model-dependent.

The iterative implementation of the cascading process was useful from a computational point of view because it avoided recursion depth limitations and allowed the simulation of larger cascades. This point concerns numerical stability rather than physical interpretation. The simulations do not yet include a convergence study, confidence intervals, finite-size scaling analysis, or a systematic comparison across parameters. These limitations are important and restrict the strength of the conclusions that can be drawn.

The comparison with turbulence must therefore remain qualitative. Avalanches in the toy model are dimensionless cascade events generated by local stochastic rules. They are not velocity gradients, eddies, energy fluxes, or viscous dissipation events in the Navier–Stokes sense. The fact that the avalanche distribution has a broad tail may be formally reminiscent of intermittent turbulent activity, but it does not imply that the toy model and turbulent flows share the same physical mechanism or the same universality class.

An intermittency-like diagnostic can be computed from the avalanche-size sample. If S_i denotes the avalanche size in the i-th event, one may define the sample flatness

$$F_S = \frac{\left\langle (S - \langle S \rangle)^4 \right\rangle}{\left\langle (S - \langle S \rangle)^2 \right\rangle^2}. \tag{22}$$

In the present simulation, this quantity takes an approximate value

$$F_S \approx 5.39.$$

Since a Gaussian variable has flatness equal to 3, the value $F_S > 3$ suggests a distribution with enhanced tails relative to a Gaussian distribution. This provides an intermittency-like indicator within the toy model. It should not be confused, however, with the flatness of velocity derivatives in turbulence, which is a physically different observable derived from the velocity field.

4.4. Avalanche-Based Second-Order Scaling

In turbulence, the second-order structure function is defined from velocity increments and satisfies

$$S_2(\ell) \sim (\varepsilon \ell)^{2/3}$$

in the inertial range [8]. The toy model does not contain a velocity field, a spatial velocity increment, or a physical energy dissipation rate. Therefore, a genuine turbulent structure function cannot be computed from the avalanche data.

Instead, one may define an avalanche-based second-order statistic as a purely internal diagnostic of the toy model. For instance, if $A(n)$ denotes the avalanche size recorded at event index n, one may consider

$$Q_2(r) = \left\langle |A(n+r) - A(n)|^2 \right\rangle, \tag{23}$$

where r is an event-index separation. Alternatively, if spatial information about avalanche clusters is retained, one may define a second-order statistic based on cluster-size fluctuations over spatial separation. In either case, the resulting quantity is an avalanche-based diagnostic, not a Navier–Stokes structure function.

The plot in Figure 3 should therefore be interpreted as an exploratory scaling plot for an avalanche-based second-order statistic. A linear trend on a log–log scale indicates that the statistic may exhibit scale-dependent behavior over the observed range. It does not demonstrate Kolmogorov scaling, nor does it establish self-similar energy transfer in the physical sense used in turbulence theory.

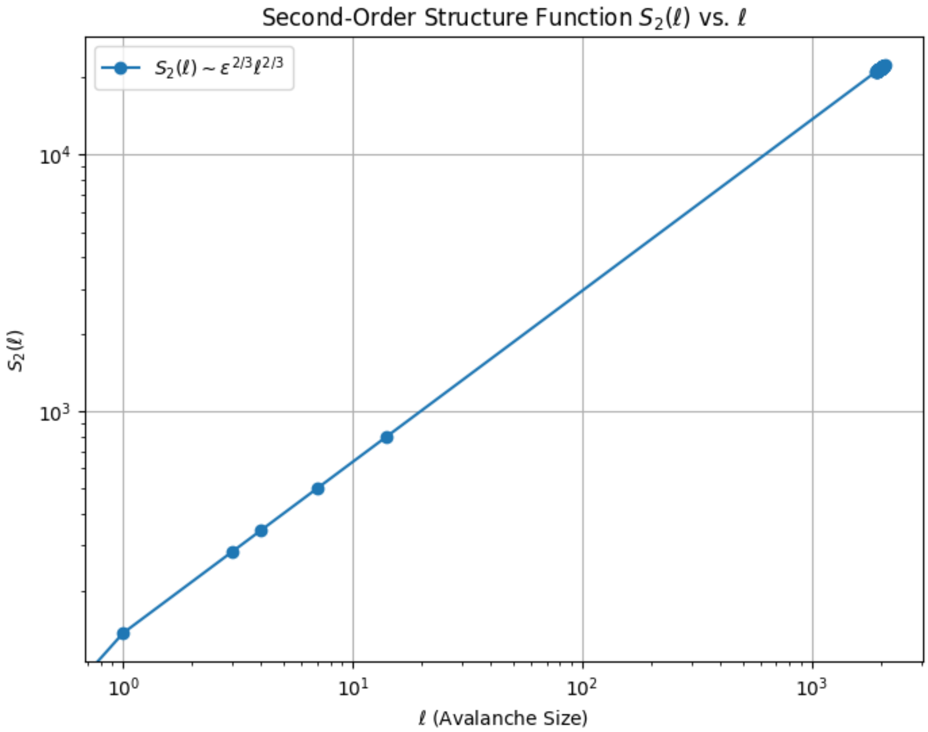

Figure 3. Exploratory log–log plot of an avalanche-based second-order statistic. The figure is used to assess scale-dependent behavior within the toy model. It should not be interpreted as a Navier–Stokes structure function or as evidence of Kolmogorov inertial-range scaling.

The observed scaling tendency is consistent with the broader interpretation that the toy model produces multiscale avalanche dynamics. However, the analogy with turbulent structure functions remains formal. In turbulence, structure functions are computed from velocity increments and are connected to the energy cascade through the Navier–Stokes equations. In the toy model, the corresponding statistic is constructed from avalanche data and has no direct dimensional interpretation as kinetic-energy transfer.

4.5. Mean Avalanche Size and Dimensional Analogy

The mean avalanche size in the toy model is

$$\langle S \rangle = \frac{1}{N_a} \sum_{i=1}^{N_a} S_i, \tag{24}$$

where N_a is the number of recorded avalanches and S_i is the size of the i-th avalanche. For the simulation reported here, we obtained

$$\langle S \rangle \approx 96.22.$$

This value is a dimensionless mean event size. It should not be identified with the physical turbulent dissipation rate ε, whose dimensions are those of energy per unit mass and unit time, namely $\mathrm{m}^2/\mathrm{s}^3$.

In Kolmogorov phenomenology, dimensional reasoning gives the estimate

$$\varepsilon \sim \frac{U^3}{L}, \tag{25}$$

where U is a characteristic velocity and L is an integral length scale. This relation is dimensionally meaningful for fluid turbulence because U and L have physical units. By contrast, the value $\langle S \rangle \approx 96.22$ is a count of removed tiles and has no intrinsic physical dimension. Therefore, equating

$$\frac{U^3}{L} \sim \langle S \rangle$$

would require an additional calibration factor with physical units. Without such a calibration, the operation is dimensionally incomplete.

For this reason, the present paper does not infer a physical velocity, Reynolds number, Kolmogorov microscale, or energy spectrum from $\langle S \rangle$. Such calculations may be useful as illustrative dimensional exercises only after an explicit mapping between tile removals and physical energy dissipation has been defined. Since no such mapping is part of the current model, the numerical value $\langle S \rangle \approx 96.22$ is used only to characterize the internal avalanche statistics of the toy model.

Assessing the Validity of the SOC–Turbulence Analogy

The numerical results indicate that the toy model can generate broad avalanche-size distributions and intermittency-like statistics. These features motivate a formal comparison with turbulence, where multiscale activity, intermittency, and scaling laws are also central. However, the analogy has clear limitations.

First, the toy model operates in a dimensionless lattice framework, whereas the Navier–Stokes equations describe physical fields with units, conservation laws, pressure coupling, viscous stresses, and incompressibility. The avalanche size S is not a physical energy dissipation rate.

Second, the fitted exponent $\tau \approx 1.32$ is an effective exponent obtained from one finite simulation. It should not be directly compared with exponents associated with turbulent structure functions, dissipation statistics, or intermittency corrections. In particular, the present data do not support the claim that the toy model and turbulent flows belong to the same universality class.

Third, the numerical analysis is limited. A stronger statistical validation would require simulations with multiple lattice sizes, multiple random seeds, confidence intervals for the fitted exponent, finite-size scaling, convergence tests, and comparison with alternative heavy-tailed distributions. These points are left for future work.

5. Conclusions

In this work, we explored a limited and exploratory connection between self-organized criticality (SOC) in a discrete toy model and some statistical features commonly discussed in turbulence phenomenology. The toy model generates avalanche-like events through local stochastic interactions, and the numerical results show a broad avalanche-size distribution that can be fitted, over the observed range, by an effective power-law exponent. This behavior is compatible with SOC-like phenomenology in the sense that small events are frequent while larger events remain possible.

The comparison with turbulence was formulated as a qualitative and statistical analogy. Turbulent flows are governed by the Navier–Stokes equations and involve velocity fields, pressure, incompressibility, nonlinear momentum transport, and viscous dissipation. The toy model does not contain these physical ingredients. Therefore, avalanche sizes in the model cannot be identified with turbulent energy dissipation events in a direct physical sense. They may only be compared with turbulent observables at the level of broad distributions, intermittent activity, and multiscale behavior.

The avalanche-based second-order statistic considered in the numerical section suggests scale-dependent behavior within the toy model. However, it should not be interpreted as a Navier–Stokes structure function or as evidence of Kolmogorov inertial-range scaling. Similarly, the mean avalanche size is a dimensionless quantity and cannot be used, without an additional dimensional calibration, to infer a physical energy dissipation rate, Reynolds number, Kolmogorov microscale, or energy spectrum.

The main contribution of the paper is therefore conceptual. It provides a minimal stochastic lattice framework in which SOC-like avalanche statistics can be examined and cautiously compared with selected features of turbulent cascade phenomenology. The results do not establish that SOC and turbulence share the same underlying mechanism or belong to the same universality class. Rather, they suggest that SOC-inspired toy models may be useful as simplified exploratory tools for understanding multiscale activity, heavy-tailed statistics, and intermittency-like behavior.

Several limitations remain. The numerical analysis was performed for a fixed lattice size, a fixed triggering probability, and a finite number of time steps. A stronger validation would require simulations across multiple lattice sizes, different parameter values, independent random seeds, confidence intervals for the fitted exponents, finite-size scaling analysis, and comparison with alternative heavy-tailed distributions. Future work should also clarify whether a more physically grounded mapping can be constructed between avalanche activity in discrete models and energy-transfer or dissipation mechanisms in fluid flows.

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