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Curvature–Matter Feedback in Semiclassical Cosmology: A Reduced Mathematical Framework for Geometry-Induced Quantum Matter Production

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Abstract

Quantum fields on a time-dependent geometry can undergo particle production, but this does not establish a self-amplifying response of the geometry. We study a conditional two-amplitude closure for a signed curvature perturbation and a signed renormalized stress response. The physical assumptions are specified through constrained projection, a low-frequency response expansion, and a stable single-pole approximation; none is asserted to hold for every semiclassical state. A quantitative memory-remainder estimate and a conditional scalar-field example make the scope of this approximation explicit. For the resulting cooperative reaction–diffusion system, a weighted spectral argument identifies the first loss of stability with the homogeneous mode. For the homogeneous cubic completion, a coercive gradient potential proves global existence and convergence to an equilibrium, including asymptotic stability at the critical parameter. We also establish the local spatial bifurcation, distinguish quintic corrections from a degenerate bifurcation, and classify a constant symmetry-breaking perturbation. Renormalization of the cosmological constant, conservation, higher-curvature terms, and the distinction between physical and effective-theory runaway solutions are treated explicitly. The contribution is a response-level diagnostic and its mathematical limitations, rather than a new microscopic theory or an observational cosmological prediction.

Keywords: semiclassical gravity; renormalized stress response; curvature–matter feedback; reaction–diffusion equations; bifurcation; memory effects

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1. Introduction

Particle creation in expanding universes, black-hole radiation, and de Sitter thermal effects illustrate the dependence of quantum fields on spacetime geometry [1–6]. Semiclassical gravity addresses a different issue: the renormalized stress expectation value is returned to the geometric equation [7–10]. A third issue is whether this reciprocal response destabilizes a particular background. These three statements are logically distinct. In particular, energy transferred to quantum fields can damp the perturbation producing it, as in the anisotropy-damping calculations of Hu and Parker [11] and the dissipative response analyzed by Hu and Sinha [12].

The question considered here is whether a specified local two-mode response possesses a growing eigenmode and whether a dissipative nonlinear completion prevents

unbounded amplitude growth. The underlying matrix criterion and the pitchfork normal form are standard. The purpose of the paper is to organize these tools into an explicitly conditional semiclassical-response diagnostic, to state the assumptions required to use it, and to distinguish its conclusions from statements about a complete spacetime. There is no claim of a new theory of quantum gravity, a universal instability of the vacuum, or a new observational prediction.

Several existing calculations delimit this question. Higher-curvature cosmology can have additional dynamical modes [13,14]; semiclassical linear response has nonlocal, state-dependent kernels [15]; and stress fluctuations require information beyond the mean equation [16]. Capanelli et al. [17] exhibit runaway gravitational production for nonminimally coupled dark photons. That matter-mode mechanism is not a derivation of the reciprocal closure considered here. Likewise, viscosity-driven cosmological evolution [18] and periodic evolution in matter–geometry coupled gravity [19] concern specified cosmological dynamics rather than the stability of an abstract two-pole response.

The analysis establishes three linked results: the exact linear threshold on a fixed bounded domain, the global homogeneous cubic dynamics, and the local spatial bifurcation. The accompanying propositions quantify a memory expansion and treat higher-order saturation and symmetry breaking. Sections 2–4 specify physical assumptions; Sections 5–7 contain the deterministic analysis; Sections 8 and 9 delimit memory, noise, and cosmological interpretation. The dimensionless gain introduced in Section 3 describes the product of the static reciprocal response factors.

2. Semiclassical Response, Renormalization, and Conservation

2.1. Quantum Fields on a Time-Dependent Geometry

Consider a real scalar field ϕ on a four-dimensional globally hyperbolic spacetime with signature $(-, +, +, +)$ and action

$$S_\phi = -\frac{1}{2} \int \sqrt{-g} \left[g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + (m^2 + \xi R) \phi^2 \right] d^4x. \quad (1)$$

Variation with respect to ϕ gives $(\square - m^2 - \xi R)\phi = 0$. For a spatially flat FLRW metric, write the scale factor as $A(\eta)$ to distinguish it from a damping coefficient. Then

$$ds^2 = A^2(\eta)(-d\eta^2 + d\mathbf{x}^2), \quad \chi_k'' + \omega_k^2 \chi_k = 0, \quad \omega_k^2 = k^2 + A^2 m^2 + (6\xi - 1) \frac{A''}{A}, \quad (2)$$

where $\chi = A\phi$ and $R = 6A''/A^3$. When initial and final positive-frequency bases are specified, the Bogoliubov coefficients obey $|\alpha_k|^2 - |\beta_k|^2 = 1$, with occupation number $|\beta_k|^2$ for the initial vacuum [2,20]. The stress tensor is more fundamental than a particle number in a general nonstationary spacetime.

2.2. Renormalized Gravitational Couplings

Use units $\hbar = c_{\text{light}} = 1$. At the curvature-squared order, a convenient local gravitational action is

$$S_g^{\text{ren}} = \int \sqrt{-g} \left[\frac{R - 2\Lambda_R}{16\pi G_R} + \alpha_R R^2 + \beta_R R_{\mu\nu} R^{\mu\nu} \right] d^4x. \quad (3)$$

In four dimensions an Euler-density term does not change the local bulk equations, and total derivatives require separate boundary treatment. Define

$$H_{\mu\nu}^{(1)} := \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} R^2 d^4x, \quad H_{\mu\nu}^{(2)} := \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} R_{\alpha\beta} R^{\alpha\beta} d^4x. \quad (4)$$

With this convention the renormalized semiclassical equation reads

$$G_{\mu\nu} + \Lambda_R g_{\mu\nu} + 16\pi G_R (\alpha_R H_{\mu\nu}^{(1)} + \beta_R H_{\mu\nu}^{(2)}) = 8\pi G_R \langle T_{\mu\nu} \rangle_{\text{ren}}. \quad (5)$$

All couplings in this equation are finite renormalized parameters, fixed at a fixed renormalization scale. In particular, Λ_R is renormalized. For a massive field the local divergent terms in a regulated effective action include structures proportional to m^4 , $m^2 R$, and curvature-squared invariants. They renormalize the vacuum term, Newton's coupling, and the quadratic couplings, respectively [8,10,21]. Splitting a constant vacuum stress between the right-hand side and Λ_R is a convention; it must not be counted twice. Renormalization does not solve the small observed cosmological-constant problem.

For a free quantum field on a classical four-dimensional metric, the local ultraviolet counterterms relevant at this order do not require an independent divergent R^n term for every n . Quantizing gravitational fluctuations is a different problem: perturbative Einstein gravity requires an effective action with progressively higher-dimension operators. In gravitational effective field theory, their renormalized coefficients are finite inputs, and calculations are organized to a fixed order in curvature and derivatives below a cutoff [22]. A truncated action is not an instruction to trust every solution at arbitrarily high frequency.

2.3. Linearization and the Local Part of the Response

Let $g^{(0)}$ be a background solution with an admissible Hadamard state and let $h_{\mu\nu} = \delta g_{\mu\nu}$. Renormalization conditions are held fixed during a metric variation. Linearizing (5) gives

$$\delta G_{\mu\nu} + \Lambda_R h_{\mu\nu} + 16\pi G_R (\alpha_R \delta H_{\mu\nu}^{(1)} + \beta_R \delta H_{\mu\nu}^{(2)}) = 8\pi G_R \delta \langle T_{\mu\nu} \rangle_{\text{ren}}. \quad (6)$$

Thus the cosmological constant enters the linearized geometric operator explicitly. One does not introduce a fluctuating $\delta \Lambda_R$ when perturbing the metric at fixed theory. Changing the renormalization scale, with the corresponding running of all terms, is a different operation [23,24].

A causal response can be written schematically as

$$\delta \langle T_{\mu\nu}(x) \rangle_{\text{ren}} = L_{\mu\nu}^{\text{loc}}[h](x) + \int \Pi_{\text{ret}\mu\nu}^{\alpha\beta}(x, x') h_{\alpha\beta}(x') dV_{x'} + \delta T_{\mu\nu}^{\text{state}}(x). \quad (7)$$

Here the normalization of Π_{ret} includes the conventional response factors, L^{loc} includes contact terms and the chosen local renormalization contributions, and δT^{state} records changes in state preparation. Moving finite conserved local terms between the two sides of (6) changes this split but not the full equation. In particular, the kernel alone is not the entire renormalized metric variation [15,25].

Algebraic nonlinearity in curvature does not invalidate linear response. For example,

$$(R_0 + \delta R)^n = R_0^n + n R_0^{n-1} \delta R + O((\delta R)^2). \quad (8)$$

The variation of the corresponding Euler–Lagrange tensor also contains derivatives of h . These must be retained to the chosen effective-theory order. Linear response requires small perturbations and a controlled background/derivative regime; it does not require the action to be linear in curvature.

2.4. Damping and the Conservation Constraint

A concrete negative-feedback effect is the damping of anisotropy by quantum particle production in the scalar-field Bianchi-I models of Hu and Parker [11]. Hu and Sinha [12] re-

late the dissipative anisotropy response to the noise kernel through an influence functional. These are examples of damping of a geometric degree of freedom, not calculations of a negative value of the particular coefficient c defined below. A dissipative, frequency-dependent tensor response cannot in general be assigned the sign of a single scalar coefficient.

Diffeomorphism invariance and the field equations imply

$$\nabla_\mu \langle T^{\mu\nu} \rangle_{\text{ren}} = 0, \quad \delta(\nabla_\mu \langle T^{\mu\nu} \rangle_{\text{ren}}) = 0. \quad (9)$$

The second identity includes variations of the connection and of index raising; it is not simply the background divergence of an arbitrarily prescribed δT . A valid projection must respect it and the gravitational constraints. Calling a scalar variable a “complete stress response” does not by itself prove that arbitrary reduced coefficients have a covariant realization.

For illustration, a covariant two-scalar system with potential

$$V(\psi, \varphi) = V_0 + \frac{A_0}{2} \psi^2 + \frac{D_0}{2} \varphi^2 - B_0 \psi \varphi + \frac{\lambda_0}{4} \psi^4, \quad A_0, D_0, \lambda_0 > 0, \quad (10)$$

satisfies, on a homogeneous FLRW solution,

$$\ddot{\psi} + 3H\dot{\psi} + V_\psi = 0, \quad \ddot{\varphi} + 3H\dot{\varphi} + V_\varphi = 0, \quad \dot{\rho} + 3H(\rho + p) = 0, \quad \rho + p = \dot{\psi}^2 + \dot{\varphi}^2. \quad (11)$$

Choose $V_0 > 0$ to obtain an expanding stationary background. In a slow overdamped sector, neglecting accelerations gives rates $a = A_0/(3H_0)$, $d = D_0/(3H_0)$, and $b = c = B_0/(3H_0)$ after compatible amplitude normalization. The product of the reciprocal responses can exceed the product of the damping rates when $B_0^2 > A_0 D_0$. The exact covariant model then has a saddle of the potential; growth releases potential energy while its full stress tensor remains conserved. Close to this threshold the unstable curvature of the potential can be small relative to H_0^2 , as required for the slow unstable mode. This example establishes that conservation does not prohibit cooperative instability. It does *not* identify ψ with spacetime curvature or supply a quantum-field realization of the closure. Such an identification requires a specified scalar–tensor model and a separate reduction.

More generally, if a model splits stress into exchanging components, source terms must cancel in the total balance. No independent positive creation source is appended in the present work. The theorems below are conditional on obtaining a constraint-compatible projected system; they do not certify that every parameter choice satisfies (9) microscopically.

3. Projection and the Local Response Closure

3.1. Explicit Amplitudes and Normalization

Fix a background foliation Σ_t , an identification of perturbed and background events, and a complete gauge or a relational clock. Let $W_\ell(t; x, y)$ be a smooth spatial averaging kernel with $\int_{\Sigma_t} W_\ell(t; x, y) dV_0(y) = 1$. For fixed positive reference scales R_* and T_* , one possible scalar choice is

$$C(t, x) = R_*^{-1} \int_{\Sigma_t} W_\ell(t; x, y) \delta R_{\text{gf}}(t, y) dV_0(y),$$

$$M(t, x) = T_*^{-1} \int_{\Sigma_t} W_\ell(t; x, y) \delta(g^{\mu\nu} \langle T_{\mu\nu} \rangle_{\text{ren}})_{\text{gf}}(t, y) dV_0(y). \quad (12)$$

Both amplitudes are dimensionless and signed. In the second line, varying the metric contraction is essential. An energy-density or anisotropic-stress projection could be used instead, but would define a different closure and different coefficients. The averaging operation here is defined on the background and is applied to the perturbation. A time-

dependent physical window generates additional terms on differentiation, which must be retained or included in the approximation error.

The label gf specifies the chosen gauge; δR is not automatically gauge invariant when the background curvature changes with time. For example, one may use a fully fixed scalar cosmological gauge for nonzero modes and a prescribed proper-time clock and initial slice for the homogeneous sector. Equivalently, gauge-invariant relational perturbations may replace the gauge-fixed quantities. The Hamiltonian and momentum constraints must first be solved, or propagated in the full perturbation system, before testing closure on these two observables. Eliminating all other variables generally introduces memory and dependence on their initial data. The two projected observables need not span an invariant sector of an arbitrary semiclassical model.

3.2. What the Local Approximation Assumes

For the deterministic autonomous closure we impose the following modelling assumptions, separately from the mathematical hypotheses of the theorems.

- (A1) A two-observable constrained scalar sector can be isolated over the time and spatial band of interest. State preparation and eliminated initial data supply no unresolved persistent forcing.
- (A2) The nonlocal remainder after local contact terms are separated has sufficiently short memory, or is approximated by a stable relaxation pole with a quantitatively small residual on that band. A low frequency alone does not imply a dissipative pole.
- (A3) A quasi-stationary background allows approximately constant coefficients. A sufficient hierarchy is $\tau_{\text{fast}} \ll T_{\text{dyn}} \ll T_{\text{bg}}$, where T_{bg} measures fractional variation of the background and coefficients. This is a substantive restriction on an expanding cosmology. Near a critical point, slowing of the retained mode can violate $T_{\text{dyn}} \ll T_{\text{bg}}$; the autonomous bifurcation is then only a frozen-parameter statement, not a theorem about passage through that point in an evolving universe.
- (A4) The spatial response is analytic and isotropic to quadratic order in the physical wavenumber. Its leading correction gives positive diagonal smoothing coefficients in the chosen variables. Cross-diffusion, nonanalytic terms, and extra slow modes are negligible in the retained band.
- (A5) Perturbations, frequencies, and curvature remain within the effective-theory cutoff and the range of the amplitude expansion. Any high-frequency higher-derivative modes discarded by order reduction are consistently excluded from the initial data.

These assumptions are based on the projection and response methods [26,27], not derived here for a particular quantum state. In a patch where a physical smoothing length is meaningful, a possible spatial hierarchy is $\ell_{\text{micro}} \ll \ell \ll L_{\text{mode}}$. In comoving FLRW coordinates the spatial operator would carry $A(t)^{-2}$; freezing this factor is part of (A3). The fixed Euclidean domain of the theorems is a model for the spatially resolved amplitudes, not a boundary of the Universe.

Under these assumptions the model is

$$\partial_t C = D_C \Delta C - aC + bM - \mu C^3, \quad \partial_t M = D_M \Delta M + cC - dM. \quad (13)$$

Here $a, d > 0$ are inverse relaxation times; b, c are reciprocal response rates; and $D_C, D_M > 0$ have dimensions length squared/time. With dimensionless amplitudes, b, c, μ also have dimensions inverse time. The cubic has $\mu > 0$ in the nonlinear analysis and is absent in the linear theorem. Its dissipative sign and the absence of quadratic terms are additional assumptions, not implications of semiclassical gravity.

For the cooperative sector $b, c > 0$ define

$$\mathcal{G} := \frac{bc}{ad}, \quad r := \frac{bc}{d} - a = a(\mathcal{G} - 1). \quad (14)$$

We use G_R for Newton's constant and $\mathcal{G}$ for the gain. If $C' = uC$ and $M' = vM$ with $u, v > 0$, then $b' = ub/v$ and $c' = vc/u$, so bc and $\mathcal{G}$ are normalization invariant. Individual coefficients are not. At zero frequency the two transfer factors of the linear closure are b/a and c/d , whose product is the gain. It compares static reciprocal susceptibilities, not the amount of produced matter or a curvature scale by itself.

3.3. A Quantitative Memory Expansion

The following elementary estimate specifies one part of the approximation. It does not apply directly to an unsmeared distributional stress kernel.

Proposition 1 (Moment expansion and its limitations). *Let a scalar regularized causal kernel satisfy $\int_0^\infty (1 + s^2)|K(s)| ds < \infty$, and let $C \in C^2(\mathbb{R})$ have bounded second derivative. Define*

$$M_K(t) = \int_0^\infty K(s)C(t-s) ds, \quad k_j = \int_0^\infty s^j K(s) ds \quad (j = 0, 1).$$

Then

$$|M_K(t) - k_0 C(t) + k_1 \dot{C}(t)| \leq \frac{1}{2} \|\ddot{C}\|_\infty \int_0^\infty s^2 |K(s)| ds. \quad (15)$$

If $k_0 \neq 0$ and $\tau = k_1/k_0 > 0$, the stable-pole response $\hat{K}_p(z) = k_0/(1 + \tau z)$ matches the first two low-frequency moments. It is realized by $\tau \dot{M} + M = k_0 C$ with a compatible prepared history.

Proof. Taylor's formula with integral remainder gives $C(t-s) = C(t) - s\dot{C}(t) + \int_0^s (s-v)\ddot{C}(t-v) dv$. Multiplication by $K(s)$ and absolute integration prove (15). For $\text{Re } z \geq 0$, the same integral remainder for e^{-zs} implies

$$\hat{K}(z) = k_0 - k_1 z + \mathcal{R}(z), \quad |\mathcal{R}(z)| \leq \frac{1}{2} |z|^2 \int_0^\infty s^2 |K(s)| ds.$$

Also $k_0/(1 + \tau z) = k_0 - k_1 z + k_0 \tau^2 z^2/(1 + \tau z)$. Thus, in this half-plane, the difference of the two responses is bounded by $|z|^2 [\frac{1}{2} \int s^2 |K| + |k_0| \tau^2]$. The pole formula follows by taking the Laplace transform of the relaxation equation, or by using the kernel $(k_0/\tau)e^{-s/\tau}$ for a prepared history. $\square$

If the memory starts at a finite initial time, the missing past produces a transient; it cannot be dropped without preparation or a transient error estimate. Moment matching is absolute, not relative, when k_0 is small. It also gives no stable-pole realization if $k_1/k_0 \leq 0$. A massive vacuum below a pair-production threshold can have predominantly dispersive analytic response; a massless field can have long tails and nonanalytic terms. Neither case is automatically a one-relaxation-time medium. Even a small response error can matter near a zero of the dispersion determinant; reliable classification requires an error smaller than the relevant spectral margin.

3.4. Spatial Smoothing and Causal Scope

A retained scalar pole of the form

$$\hat{\chi}_M(z, k) \simeq \frac{c}{z + d + D_M |k|^2} \quad (16)$$

corresponds to the second equation in (13) without its geometric partner. The term $D_M \Delta M$ is the leading spatial correction to a relaxation pole and therefore acts as smoothing. These are two descriptions of the same conditional gradient expansion. A numerator correction proportional to $|k|^2$ would instead produce a term involving ΔC ; the diagonal-diffusion choice omits it and must be tested. Positivity of D_M is not a consequence of covariance. The same cautions apply to D_C .

A parabolic equation has instantaneous spatial tails. It is not a fundamental relativistic propagation law and is not valid at arbitrary wavenumber or arbitrarily short time. A covariant parent model retains causal propagation and often second time derivatives. Diffusion describes only a coarse-grained low-frequency branch in a preferred background frame. Consequently, while the PDE theorem treats its own full Laplacian spectrum exactly, only the retained long-wavelength modes can be assigned a microscopic interpretation.

4. A Conditional FLRW Scalar-Field Example

A concrete starting point is (1) on a chosen FLRW background in a Hadamard state. Solving (2) and computing the renormalized stress with a consistent subtraction scheme is a standard route [21,28]. The field equation alone does not determine a relaxation time. One must vary both the modes and the renormalized subtraction/contact terms, apply (12), and compute the scalar retarded susceptibility in the constrained sector.

Suppose, conditionally, that after freezing the background this computed response admits

$$\hat{\chi}_M(z, k) = \frac{\chi_0}{1 + \tau z + \ell_M^2 |k|^2} + \mathcal{E}(z, k), \quad \tau > 0, \quad \ell_M^2 > 0, \quad (17)$$

with a small quantified remainder on the chosen band. The pole part yields, exactly,

$$\tau \partial_t M + M - \ell_M^2 \Delta M = \chi_0 C, \quad d = \tau^{-1}, \quad c = \chi_0 / \tau, \quad D_M = \ell_M^2 / \tau. \quad (18)$$

If the projected geometric transfer is likewise $\hat{\chi}_C = b / (z + a + D_C |k|^2)$, the static threshold is $b\chi_0 = a$. This is an explicit coefficient dictionary and a conditional worked realization of the approximation, not a calculation of χ_0, τ, ℓ_M for a free scalar vacuum. The sign of χ_0 and the existence of a pole remain to be established from that calculation. A free field has no generic collisional relaxation mechanism that guarantees (17).

The geometric approximation can itself fail. In the classical $f(R) = R + \alpha R^2 - 2\Lambda_R$ illustration, the trace equation with conventional matter stress $T = g^{\mu\nu} T_{\mu\nu}$ is

$$6\alpha \square R - R + 4\Lambda_R = 8\pi G_R T. \quad (19)$$

Around a constant-curvature background, $\alpha > 0$ gives $m_s^2 = 1/(6\alpha) > 0$ and

$$\delta \ddot{R} + 3H_0 \delta \dot{R} - A^{-2} \Delta \delta R + m_s^2 \delta R = -\frac{4\pi G_R}{3\alpha} \delta T. \quad (20)$$

Only on a slow overdamped branch with $|\partial_t| \ll 3H_0$ can the acceleration be neglected. For the homogeneous constant- H_0 sector this gives $a = m_s^2 / (3H_0)$. Formally the retained spatial term has coefficient $1/(3H_0)$ in front of the physical Laplacian, or $A(t)^{-2}/(3H_0)$ in comoving coordinates. Over an overdamped evolution longer than a Hubble time, freezing $A(t)$ is not justified; physical coordinates instead introduce expansion transport. Thus this scalaron example motivates a homogeneous relaxation rate but does not derive the autonomous spatial model. For the particular trace projection in (12), the resulting b is negative with these sign conventions. Reorienting M changes both reciprocal signs and leaves their product unchanged. This example demonstrates why neither two first-order

equations nor cooperative signs are universal. The phenomenological theorem is applied only when a suitable projection actually yields its hypotheses.

There is also a useful exact limiting check. For a massless conformally coupled scalar, $m = 0$ and $\xi = 1/6$, (2) becomes $\chi_k'' + k^2\chi_k = 0$. In the conformal vacuum on spatially flat FLRW there is no mixing of these conformal modes. Nevertheless the renormalized stress can have a trace anomaly [29]; absence of particle production does not imply vanishing complete stress response. A calculation based only on particle number would therefore miss a relevant part of M .

5. The Linear Reaction–Diffusion Threshold

Let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$, be bounded and connected with C^2 boundary. Impose homogeneous Neumann conditions

$$\partial_\nu C = \partial_\nu M = 0 \quad \text{on } \partial\Omega, \quad (21)$$

where ν is the outward unit normal. These conditions preserve the constant mode; a periodic domain has the same first threshold. Consider (13) with $\mu = 0$ on $L^2(\Omega)^2$.

Theorem 1 (Exact threshold for the local linear closure). *Assume $a, d, D_C, D_M, b, c > 0$. The Neumann problem generates a unique analytic semigroup on $L^2(\Omega)^2$. If $bc < ad$, every solution decays exponentially, with*

$$\|C(t)\|_2^2 + \|M(t)\|_2^2 \leq \frac{\max\{b, c\}}{\min\{b, c\}} e^{-2\omega t} (\|C(0)\|_2^2 + \|M(0)\|_2^2), \quad \omega = \frac{a + d - \sqrt{(a - d)^2 + 4bc}}{2} > 0. \quad (22)$$

If $bc = ad$, the only neutral eigenspace consists of spatial constants proportional to (b, a) ; all complementary modes decay exponentially. If $bc > ad$, the homogeneous mode has a positive real eigenvalue and arbitrarily small data along this mode grow exponentially.

Proof. The diagonal Neumann Laplacians with domain $H_N^2(\Omega)^2$ generate analytic semigroups. The reaction matrix is a bounded perturbation. Alternatively, the similarity below constructs the semigroup from a self-adjoint operator bounded above.

Write $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$ for the eigenvalues of $-\Delta_N$, with real orthonormal eigenfunctions e_j and $e_0 = |\Omega|^{-1/2}$. Connectedness makes λ_0 simple. The coefficients of each eigenfunction solve the matrix equation with

$$A_j = \begin{pmatrix} -a - D_C\lambda_j & b \\ c & -d - D_M\lambda_j \end{pmatrix}, \quad \text{tr } A_j = -a - d - (D_C + D_M)\lambda_j, \quad (23)$$

$$\det A_j = ad - bc + (aD_M + dD_C)\lambda_j + D_CD_M\lambda_j^2. \quad (24)$$

The projection is justified by Green's identity: the boundary integral in $\int_\Omega (\Delta C)e_j$ vanishes by $\partial_\nu C = 0$, and the second integration by parts uses $\partial_\nu e_j = 0$ to give $-\lambda_j \int_\Omega C e_j$.

For $S = \text{diag}(\sqrt{c}, \sqrt{b})$,

$$B_j = S A_j S^{-1} = \begin{pmatrix} -a - D_C\lambda_j & \sqrt{bc} \\ \sqrt{bc} & -d - D_M\lambda_j \end{pmatrix} = B_0 - \lambda_j \text{diag}(D_C, D_M). \quad (25)$$

Thus all eigenvalues are real. If $bc < ad$, B_0 is negative definite and its largest eigenvalue is $-\omega$ from (22). Its Rayleigh quotient gives $z^T B_j z \leq -\omega |z|^2$, uniformly in j .

For compatible H^2 data, put $X = \sqrt{c} C$, $Y = \sqrt{b} M$, and $E = \frac{1}{2}(\|X\|_2^2 + \|Y\|_2^2)$. Direct differentiation and integration by parts yield

$$\begin{aligned} E' &= -D_C \|\nabla X\|_2^2 - D_M \|\nabla Y\|_2^2 + \int_{\Omega} (-aX^2 - dY^2 + 2\sqrt{bc}XY) dx \\ &\leq -2\omega E. \end{aligned} \quad (26)$$

In this calculation, $\int_{\partial\Omega} X \partial_\nu X$ and its Y counterpart are zero. Integration of (26) and equivalence of the weighted and unweighted norms prove (22). Finite spectral truncations of arbitrary L^2 data obey the same estimate. Applying it to differences of truncations gives convergence in $C([0, T]; L^2(\Omega)^2)$, and hence the estimate for the unique mild solution.

At equality, $A_0(b, a)^T = 0$ and the other homogeneous eigenvalue is $-(a + d)$. For $j \geq 1$, (25) is bounded above by $-\lambda_1 \min(D_C, D_M)I$. This proves exponential decay off the one-dimensional neutral space. If $bc > ad$, $\det A_0 < 0$ and the larger eigenvalue

$$\lambda_+ = -\frac{a+d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + 4bc} \quad (27)$$

is positive. Taking initial data $\varepsilon e_0 v_+$, with v_+ an associated eigenvector, gives the exact solution $\varepsilon e^{\lambda_+ t} e_0 v_+$ for any $\varepsilon > 0$. $\square$

Remark 1 (The first mode and the unstable band). *The maximal eigenvalue decreases strictly with λ_j , by (25). The first instability is therefore homogeneous. Above threshold, a mode is unstable precisely when*

$$\lambda_j < \lambda_{\text{cut}} := \frac{-(aD_M + dD_C) + \sqrt{(aD_M + dD_C)^2 + 4D_C D_M(bc - ad)}}{2D_C D_M}. \quad (28)$$

Equality gives a neutral mode. On a fixed bounded domain, only finitely many modes can be unstable, and sufficiently close to the first threshold only the homogeneous mode is critical. There is no diffusion-driven Turing onset from a stable homogeneous state in this particular cooperative, diagonally diffusive model. Other diffusion matrices or response signs require a new calculation.

If $bc < 0$ with $a, d, D_C, D_M > 0$, every mode has negative trace and positive determinant; the cooperative symmetrization is unavailable but the claimed automatic positive-feedback instability is still absent. If one coupling is zero, a triangular system can exhibit transient forcing but cannot generate a growing feedback eigenmode. These observations concern the linear autonomous closure only.

6. Global Homogeneous Dynamics and Cubic Saturation

Set the spatial derivatives to zero and take $\mu > 0$:

$$\dot{C} = -aC + bM - \mu C^3, \quad \dot{M} = cC - dM. \quad (29)$$

The zero state is the reference perturbation state; it need not be a vacuum spacetime. Likewise a nonzero equilibrium below is a finite perturbation amplitude, not automatically a high-curvature cosmological phase.

Theorem 2 (Global convergence and the supercritical pitchfork). *Let $a, b, c, d, \mu > 0$ and define r by (14). Every initial condition in $\mathbb{R}^2$ has a unique global bounded solution of (29), and every solution converges to a single equilibrium. More precisely:*

- (i) *For $r < 0$, the origin is the unique equilibrium and is globally exponentially asymptotically stable.*

- (ii) For $r = 0$, the origin is the unique equilibrium and is globally asymptotically stable. No estimate $|z(t)| \leq Ke^{-\gamma t}|z(0)|$ with fixed $K, \gamma > 0$ holds for all sufficiently small initial states and all $t \geq 0$.
- (iii) For $r > 0$, the origin is a hyperbolic saddle and the other equilibria are

$$C_{\pm} = \pm \sqrt{r/\mu}, \quad M_{\pm} = \frac{c}{d}C_{\pm}. \quad (30)$$

Both are locally exponentially asymptotically stable. A trajectory may still converge to the origin if it belongs to its stable set; the theorem does not assert convergence of every nonzero initial condition to a nonzero branch.

With a, b, d, μ fixed and $c = d(a + r)/b$, the local bifurcation at $r = 0$ is a nondegenerate supercritical pitchfork.

Proof. The polynomial vector field is locally Lipschitz, so a unique maximal solution exists. Define the potential

$$\begin{aligned} U(C, M) &= \frac{ac}{2}C^2 + \frac{bd}{2}M^2 - bcCM + \frac{\mu c}{4}C^4 \\ &= \frac{bd}{2}\left(M - \frac{c}{d}C\right)^2 - \frac{cr}{2}C^2 + \frac{\mu c}{4}C^4. \end{aligned} \quad (31)$$

It is bounded below and coercive for every fixed r : if $|C| \rightarrow \infty$ the positive quartic dominates, and if C is bounded while $|M| \rightarrow \infty$ the square diverges. Its partial derivatives satisfy $U_C = -c\dot{C}$ and $U_M = -b\dot{M}$, so

$$\frac{d}{dt}U(C(t), M(t)) = -c\dot{C}(t)^2 - b\dot{M}(t)^2 \leq 0. \quad (32)$$

Every trajectory remains in a compact sublevel of U . The ODE continuation criterion excludes a finite maximal time.

The omega-limit set is nonempty and compact. The potential is constant on it because U decreases to a limit; invariance and (32) force every point in that set to be an equilibrium. At an equilibrium, $M = (c/d)C$ and

$$F(C; r) := rC - \mu C^3 = 0. \quad (33)$$

There are at most three equilibria, exactly as stated. The omega-limit set is connected: it is the intersection of the nested compact connected closures of the trajectory tails. A connected subset of a finite set is a singleton. Hence each solution converges to one equilibrium. This argument supplies the global convergence that cannot be inferred merely from local stability of the branches.

For the stability claims, use the positive quadratic function

$$V(C, M) = \frac{1}{2}(cC^2 + bM^2), \quad \dot{V} = -acC^2 - bdM^2 + 2bcCM - \mu cC^4. \quad (34)$$

When $bc < ad$, the same symmetric matrix as in (25) gives $\dot{V} \leq -2\omega V$. This proves global exponential attraction and stability. At $bc = ad$, with $X = \sqrt{c}C, Y = \sqrt{b}M$,

$$\dot{V} = -(\sqrt{a}X - \sqrt{d}Y)^2 - \frac{\mu}{c}X^4. \quad (35)$$

The right-hand side is strictly negative except at $(0, 0)$. Positivity of V gives Lyapunov stability, and the already proved convergence gives global attraction. Uniform local expo-

nential decay would, on differentiating the flow at the equilibrium, imply the same estimate for e^{tJ_0} . This is impossible because J_0 has a zero eigenvalue and a nonzero stationary vector.

For $r > 0$, the determinant of the Jacobian at the origin is $-dr < 0$, so it is a hyperbolic saddle. At either nonzero branch,

$$J_* = \begin{pmatrix} -a - 3\mu C_*^2 & b \\ c & -d \end{pmatrix}, \quad \det J_* = 2dr > 0, \quad \text{tr } J_* = -a - d - 3r < 0. \quad (36)$$

The matrix is similar to a symmetric one by the same positive diagonal change of variables as before. Both eigenvalues are negative, and smooth ODE linearization yields local exponential stability.

For nondegeneracy, the parameter choice in the statement gives a simple zero eigenvalue at $r = 0$, with the other eigenvalue $-(a + d)$. Its derivative with respect to r is $d/(a + d) > 0$, either from (27) or from left and right eigenvectors. A local center manifold, expressed as a graph over C , has at $r = 0$

$$M = \frac{a}{b}C + \frac{a\mu}{b(a+d)}C^3 + O(C^5). \quad (37)$$

To compute the coefficient, insert $M = (a/b)C + h_3C^3 + O(C^5)$ into the invariance equation. The cubic terms give $(a/b)(bh_3 - \mu) = -dh_3$, hence the displayed value. Including the parameter gives the reduced equation

$$\dot{C} = \frac{d}{a+d}(rC - \mu C^3) + O(r^2|C| + |r||C|^3 + |C|^5). \quad (38)$$

The parameter and cubic coefficients are nonzero and have the required opposite signs. The exact branches (30) and the stability calculation complete the pitchfork classification. $\square$

At the critical parameter, nonzero sufficiently small solutions on the center manifold satisfy

$$C(t) \sim \pm \left(\frac{2\mu d}{a+d} t \right)^{-1/2}. \quad (39)$$

Indeed (38) with $r = 0$ gives $\frac{d}{dt}C^{-2} = 2\mu d/(a+d) + O(C^2)$, and $C(t) \rightarrow 0$. Dividing its integral by t proves the asymptotic. This is a center-manifold statement; trajectories on the strong stable manifold can decay faster.

The physical content of (35) is limited and precise. At threshold the quadratic damping vanishes in one perturbation direction, but the imposed cubic produces amplitude-dependent relaxation in that direction. V and U are mathematical Lyapunov functions, not the covariant gravitational energy or a stress-energy component. Their decay neither violates (9) nor identifies a physical dissipation channel without further modelling. The cubic removes the quadratic degeneracy for nonzero perturbations, but the linearized eigenvalue remains zero.

7. Spatial Bifurcation, Higher-Order Terms, and Imperfections

7.1. What Spatial Structure Changes Locally

For the following local classical analysis take $\partial\Omega$ of class $C^{2+\alpha}$ for some $0 < \alpha < 1$; smooth compatible data suffice, and continuous data can be treated by parabolic mild solutions. The regularity assumption avoids boundary regularity issues unrelated to the bifurcation.

Theorem 3 (The first bifurcation of the nonlinear PDE). *For fixed positive a, b, d, D_C, D_M, μ and $c = d(a + r)/b > 0$, the nonlinear Neumann system (13) has a global bounded solution for every continuous initial pair, classical for positive time. Near $(C, M, r) = (0, 0, 0)$ its stationary bifurcation is exactly the homogeneous supercritical pitchfork of Theorem 2. The nonzero constant equilibria are locally exponentially stable to general sufficiently small continuous spatial perturbations for each $r > 0$. The spatial center dynamics at the first threshold has the homogeneous normal form (38).*

Proof. The diagonal heat semigroups and the locally Lipschitz polynomial reaction give a unique local mild solution on $C(\overline{\Omega})^2$; parabolic smoothing supplies its positive-time regularity [30]. Choose $R > 0$ with

$$R \geq \|C(0)\|_\infty, \quad (c/d)R \geq \|M(0)\|_\infty, \quad \mu R^2 \geq \max(r, 0).$$

Set $S = (c/d)R$. On $C = R$ and $|M| \leq S$, the first reaction is at most $rR - \mu R^3 \leq 0$; on $C = -R$ it is at least zero. On $M = S$ and $|C| \leq R$, the second reaction is at most $cR - dS = 0$, with the opposite inequality on $M = -S$. Thus the rectangle $[-R, R] \times [-S, S]$ is invariant. A first-exit argument using the scalar parabolic maximum principle, or a strict enlarged rectangle followed by a limit, proves this also with Neumann boundary conditions. The resulting uniform bound prevents finite-time loss of the mild solution.

At $r = 0$, Theorem 1 gives a one-dimensional center eigenspace consisting of constants, and a uniform negative spectral gap on its complement. The homogeneous two-dimensional subspace is invariant. A center manifold of the homogeneous ODE, with the parameter adjoined, is therefore also a local center manifold for the PDE: it has the full center tangent space and is locally invariant. The parabolic center-manifold theorem applies to the analytic generator and the smooth Nemytskii reaction on the continuous-function phase space [30]. The local reduced coefficients are consequently those in (38).

One can exclude additional nearby nonconstant equilibria directly. Write a stationary pair as its spatial mean plus a zero-mean pair. The stationary equation projected onto the zero-mean subspace has at the origin an invertible elliptic linearization, since all $j \geq 1$ blocks in (23) are strictly negative at $r = 0$. In compatible Hölder spaces the implicit-function theorem gives a unique small zero-mean part for each small mean and parameter. The zero-mean part identically zero already solves that projected equation, because the reaction of a constant is constant. Uniqueness forces it to be zero. The remaining mean equation is (33).

Finally, at either nonzero constant equilibrium the mode matrices are

$$J_{*,j} = \begin{pmatrix} -a - 3r - D_C \lambda_j & b \\ c & -d - D_M \lambda_j \end{pmatrix}. \quad (40)$$

Their symmetric similarities equal the negative-definite homogeneous block minus $\lambda_j \operatorname{diag}(D_C, D_M)$. Thus there is a strictly negative spectral bound, uniform in j . The compact parabolic semigroup on the continuous-function space has the same elliptic eigenvalues. Its exponential decay and the locally quadratic nonlinear remainder yield local exponential stability by variation of constants and a small-data bootstrap. $\square$

The theorem is local in amplitude and in the first bifurcation. For larger r , the origin can have additional unstable modes according to (28); nonconstant stationary solutions and spatial transients are not excluded. No claim of global convergence to a spatially constant state is made. On an unbounded domain the spectral gap used here closes at

long wavelength, so the bounded-domain center-manifold argument cannot simply be transferred to $\mathbb{R}^n$. The nonlinear PDE also has the gradient functional

$$\mathcal{U} = \int_{\Omega} \left[\frac{cD_C}{2} |\nabla C|^2 + \frac{bD_M}{2} |\nabla M|^2 + U(C, M) \right] dx, \quad \frac{d\mathcal{U}}{dt} = - \int_{\Omega} (cC_t^2 + bM_t^2) dx, \quad (41)$$

for classical solutions. This identity does not by itself classify all its nonconstant equilibria.

7.2. Higher-Order Odd Saturation

Replace the first reaction in (29) by $-aC + bM - \mu C^3 - \nu C^5$, where the sign of ν is stated explicitly. The stationary scalar equation is

$$C(r - \mu C^2 - \nu C^4) = 0. \quad (42)$$

Proposition 2 (Robust and degenerate odd bifurcations). *If $\mu > 0$ and $\nu \in \mathbb{R}$ is fixed, the local pitchfork at $r = 0$ remains supercritical. Its small nonzero branches satisfy*

$$C_{\pm} = \pm \sqrt{r/\mu} \left[1 - \frac{\nu}{2\mu^2} r + O(r^2) \right] \quad (r \downarrow 0). \quad (43)$$

For $\nu \geq 0$, all homogeneous trajectories remain global and bounded. If $\mu = 0$ and $\nu > 0$, the bifurcation is degenerate with $|C_{\pm}| = (r/\nu)^{1/4}$. If $\mu < 0$ and $\nu > 0$, the cubic onset is subcritical and nonzero branches have a fold at $r = -\mu^2/(4\nu)$, whenever that parameter lies in the range $c > 0$.

Proof. For $C \neq 0$ put $z = C^2 \geq 0$. The equation becomes $\mu z + \nu z^2 = r$. When $\mu > 0$, the implicit-function theorem gives $z = r/\mu - \nu r^2/\mu^3 + O(r^3)$, which yields (43). At an equilibrium,

$$\det J = d(2\mu z + 4\nu z^2). \quad (44)$$

For the small branches with $r > 0$, this is positive and the trace is negative by continuity from $-a - d$. This proves local stability and the unchanged nondegenerate cubic onset. For $\mu > 0$, $\nu \geq 0$, adding $c\nu C^6/6$ to (31) gives a coercive gradient potential, proving global boundedness as before. No such global assertion follows for $\nu < 0$.

For $\mu = 0$, direct solution gives the quarter-power branch; its determinant is $4dr > 0$ and its trace is $-a - d - 5r < 0$. For $\mu < 0$, $\nu > 0$, the parabola $r = \mu z + \nu z^2$ has its minimum at $z = -\mu/(2\nu)$, with the stated value of r . Its upper branch has $\mu + 2\nu z > 0$, so (44) is positive. Moreover its first diagonal damping coefficient is $a + 3\mu z + 5\nu z^2 = bc/d + 2z(\mu + 2\nu z) > 0$, making the trace negative. The lower branch has negative determinant and is a saddle. The small unstable branch lies on the $r < 0$ side, which is the subcritical case. If the fold lies outside $r > -a$, it is outside the cooperative parameter family under consideration. $\square$

The global sign of the highest retained odd term matters. A destabilizing quintic can invalidate boundedness even though it does not change the local cubic bifurcation. A physically derived amplitude expansion may also contain even terms and hence need not possess a pitchfork at all.

7.3. A Symmetry-Breaking Source

Expansion-dependent damping or a nonzero background cosmological constant does not, by itself, break $(C, M) \mapsto (-C, -M)$. Linear time-dependent coefficients preserve this

sign symmetry. A constant forcing or an even-order perturbation does break it. To make the issue explicit, consider

$$\dot{C} = -aC + bM - \mu C^3 + h, \quad \dot{M} = cC - dM, \quad a, b, c, d, \mu > 0. \quad (45)$$

Proposition 3 (The imperfect pitchfork). *Every trajectory of (45) is global and bounded and converges to an equilibrium. The equilibria obey*

$$M = (c/d)C, \quad F_h(C; r) = rC - \mu C^3 + h = 0. \quad (46)$$

For $r > 0$, there are three distinct real equilibria if

$$|h| < \frac{2r^{3/2}}{3\sqrt{3\mu}}. \quad (47)$$

The outer two are stable and the middle one is a saddle. Equality with $r > 0$ gives a double-root saddle-node and one simple stable equilibrium. Outside this region there is one simple stable equilibrium, except at the cusp $(r, h) = (0, 0)$, where Theorem 2(ii) applies.

Proof. The potential is $U_h = U - chC$, still coercive, and obeys the same gradient identity. The convergence proof in Theorem 2 applies because the equilibrium set remains finite. Dividing the cubic by $-\mu$ gives discriminant $4r^3/\mu^3 - 27h^2/\mu^2$; its sign yields the root count. At a root,

$$\det J = d(3\mu C^2 - r) = -dF'_h(C; r), \quad \text{tr } J = -a - d - 3\mu C^2 < 0.$$

The outer roots have $F'_h < 0$, and the middle one has $F'_h > 0$. At a double root with $r > 0$, $F'_h = 0$ and $F''_h = -6\mu C \neq 0$, so the fold is nondegenerate. The remaining simple root is stable. For a unique simple root the slope is negative, proving the final stability claim. $\square$

Thus small imperfections preserve the two stable finite branches away from the fold, while sufficiently large bias leaves one stable branch. At $r = 0$ and $h \neq 0$, $C = (h/\mu)^{1/3}$ is already nonzero; the unbiased threshold is no longer an exact onset. This describes a specified imperfection, not every possible time-dependent cosmological unfolding.

8. Memory and Stochastic Extensions

8.1. A Volterra Formulation and Its Spectral Questions

A minimal extension retaining scalar memory is

$$\begin{aligned} C_t &= D_C \Delta C - aC + b \int_0^\infty K_b(s) M(t-s) ds - \mu C^3, \\ M_t &= D_M \Delta M - dM + c \int_0^\infty K_c(s) C(t-s) ds. \end{aligned} \quad (48)$$

Initial histories on $(-\infty, 0]$, or finite-time kernels together with the omitted-history forcing, are part of the problem. The local model corresponds formally to unit point masses at $s = 0$, not to an arbitrary integrable kernel. If the regularized kernels are locally integrable, the integrated mild equations and a contraction argument provide local well-posedness on appropriate history spaces. Global bounds, decay, and a state-independent phase space require additional kernel hypotheses [31]. The original quantum kernel can be distributional, tensorial, and nonstationary; (48) is still a reduced extension.

For the linearized problem and the spatial eigenvalue λ_j , the exponential-mode determinant is

$$D_j(z) = (z + a + D_C \lambda_j)(z + d + D_M \lambda_j) - bc \hat{K}_b(z) \hat{K}_c(z). \quad (49)$$

Only in the instantaneous case is this a quadratic polynomial. Memory can introduce additional poles, branch cuts, and phase shifts. Stability analysis requires a resolvent in a specified history space and control of its inverse transform, not just the sign of a constant determinant. Algebraically decaying memory can preclude exponential decay even when no nonnegative characteristic root is present. A nonstationary FLRW kernel $K(t, s)$ need not admit a Laplace-domain criterion at all.

There is a useful restricted statement that prevents overinterpreting memory as an automatic source of oscillatory instability.

Proposition 4 (A root test for positive normalized memory). *Suppose $K_b, K_c \geq 0$ are integrable on $(0, \infty)$ and each has integral one. If $bc < ad$, no D_j in (49) vanishes for $\text{Re } z \geq 0$. If $bc > ad$, D_0 has a positive real zero. At equality, $D_0(0) = 0$.*

Proof. For $\text{Re } z \geq 0$, normalization and positivity give $|\hat{K}_i(z)| \leq 1$, whereas $|z + a + D_C \lambda_j| \geq a + D_C \lambda_j$ and $|z + d + D_M \lambda_j| \geq d + D_M \lambda_j$. For $bc < ad$ these inequalities exclude equality of the two terms in (49). If $bc > ad$, $D_0(0) = ad - bc < 0$. Along the positive real axis, D_0 is continuous and tends to $+\infty$, since the transforms are bounded by one. The intermediate-value theorem gives a positive real zero. The equality statement is direct. $\square$

This is a characteristic-root result, not an exponential-stability theorem for arbitrary histories. Sign-changing or matrix-valued quantum-response kernels need not satisfy its assumptions. A static gain by itself is then insufficient; any Nyquist-type argument must also account for open-loop singularities and the analytic domain. Moment control as in Proposition 1, tail control for history estimates, and preservation of the Ward identities are separate tasks.

8.2. Projected Stress Noise

Let $\hat{t}_{\mu\nu} = \hat{T}_{\mu\nu} - \langle \hat{T}_{\mu\nu} \rangle$. Stochastic gravity introduces a mean-zero source with covariance determined by the noise kernel

$$N_{\mu\nu\alpha\beta}(x, x') = \frac{1}{2} \langle \{ \hat{t}_{\mu\nu}(x), \hat{t}_{\alpha\beta}(x') \} \rangle. \quad (50)$$

It is a bi-distribution and requires suitable smearing, including temporal resolution when necessary. Applying constraint-compatible projection and response operators P_i gives a colored amplitude-force covariance schematically of the form

$$\mathbb{E}[\eta_i(t) \eta_j(t')] = \iint P_i^{\mu\nu}(t; x) N_{\mu\nu\alpha\beta}(x, x') P_j^{\alpha\beta}(t'; x') dV_x dV_{x'}. \quad (51)$$

The operators include the normalization and geometric conversion factors. Positivity and conservation cannot be enforced by choosing two arbitrary independent noise sources after projection. The noise and dissipation kernels originate from the same matter theory; a fluctuation–dissipation relation requires the appropriate state and stationarity assumptions [12,16].

If an additional short-correlation limit is justified for the homogeneous amplitudes, a mathematically definite approximation is the Itô system

$$dZ = F(Z) dt + \varepsilon B dW_t, \quad Z = (C, M)^T, \quad F = (-aC + bM - \mu C^3, cC - dM)^T, \quad (52)$$

where B is a specified constant matrix and W is a finite-dimensional Wiener process. This is a classical stochastic approximation to the projected noise, not an identification of quantum stress with white noise at every scale.

For completeness, it has a dissipative a priori estimate. Put $E = |Z|^2/2$ and

$$K = -a + \frac{(b+c)^2}{2d}, \quad A_* := \frac{((K+d/2)_+)^2}{4\mu}. \quad (53)$$

Young's inequality gives $Z \cdot F \leq KC^2 - (d/2)M^2 - \mu C^4 \leq -dE + A_*$. Itô's formula, first stopped on bounded balls, then the standard nonexplosion/localization argument, gives a global solution and

$$\mathbb{E}E(t) \leq e^{-dt}E(0) + \frac{A_* + \varepsilon^2 \operatorname{tr}(BB^T)/2}{d}(1 - e^{-dt}). \quad (54)$$

The localization is justified by the polynomial local Lipschitz coefficients and the coercive quadratic estimate; the latter bounds the probability of reaching radius R on a fixed time interval by a quantity tending to zero as $R \rightarrow \infty$.

In the stable linearized regime the stationary covariance is

$$\Sigma = \varepsilon^2 \int_0^\infty e^{J_0 s} BB^T e^{J_0^T s} ds, \quad J_0 \Sigma + \Sigma J_0^T + \varepsilon^2 BB^T = 0. \quad (55)$$

Additive noise does not alter the eigenvalues of the deterministic drift; it creates fluctuations around it. For the nonlinear model, however, the mean equation involves $\mathbb{E}[C^3]$, which is not generally $(\mathbb{E}C)^3$. Near-critical fluctuation growth or transitions between the stable branches are further questions, not predictions established by (55).

For a spatial stochastic model, a safe starting point is smooth spatial covariance or an appropriately trace-class covariance with sufficient smoothing for the cubic Nemytskii term. Unsmearred space-time white noise in two or three spatial dimensions does not automatically produce a classical cubic reaction-diffusion equation and can require renormalization. Infinite-dimensional well-posedness must be formulated in explicit function spaces [32]; it is not a direct consequence of Theorem 3.

9. Comparison with Cosmological and Semiclassical Models

9.1. Higher Derivatives and the Meaning of a Runaway

Curvature-squared terms have a substantial cosmological history, including anomaly-driven evolution and Starobinsky's early-universe model [13,14,29]. They need not always create an instability. For instance, (20) with $\alpha > 0$, no forcing, and a nonexpanding background has frequencies satisfying $\omega^2 = |k|^2 + m_s^2 > 0$. On an expanding constant- H_0 homogeneous background, its roots solve $z^2 + 3H_0 z + m_s^2 = 0$ and have negative real parts. This establishes a stable linear scalar sector in this example, not unconditional stability of every higher-curvature action or every background.

Higher time derivatives enlarge the formal solution space. If their coefficients are perturbative effective-theory corrections, some formal roots have frequencies outside the cutoff or are nonanalytic in the small expansion parameter. Such roots cannot be interpreted reliably within the truncated approximation. Order reduction selects the perturbative branch when justified; it is not a prescription for deleting any unwanted low-frequency instability [33]. Conversely, in an R^2 inflationary model whose scalaron lies within the intended dynamical range, that scalaron must be retained rather than removed as a spurious root.

Suen’s analysis [34] and the effective-theory and response treatments in [15,33] make clear that stability depends on the model and admissible perturbation scales. A physical growing mode within the reliable band, an artifact of extrapolating a derivative expansion, and a ghost or instability of an inappropriate fundamental action are different diagnoses. The presence of higher-curvature operators proves neither universal runaway behavior nor the failure of general relativity as a low-energy effective description [22]. No claim is made here that all problematic higher-derivative modes can always be removed consistently.

9.2. Cosmological Closure and Observational Content

In homogeneous GR with a conserved effective fluid one has

$$3H^2 = 8\pi G_R \rho + \Lambda_R, \quad \dot{\rho} + 3H(\rho + p + \Pi) = 0, \quad (56)$$

where a bulk-viscous pressure Π must be supplied by a constitutive law. The instantaneous illustrative law $\Pi = -3\zeta H$ differs from a causal relaxation law, which has its own additional variable and assumptions. Such models can be constrained through a predicted expansion history once the constitutive parameters, matter content, and data likelihood are specified. Yousaf’s viscous-fluid study [18] and periodic cosmological evolution in matter–geometry coupled gravity [19] provide relevant recent comparisons. The latter does not supply the retarded quantum-stress coefficients in (13). We invoke these studies for their stated cosmological settings, not as numerical evidence for our gain criterion.

Framework	Dynamical Input	Relation to the Present Closure
GR with effective viscosity	Friedmann equations plus an equation of state and a law for Π	Dissipation modifies an expansion history; no quantum stress-response pole follows from viscosity alone.
Matter–geometry coupled cosmology	A specified modified action and background ansatz	Conservation and exchange laws depend on that action; reduced response coefficients require its linearization.
R^2 /scalaron dynamics	Additional geometric mode with mass and kinetic evolution	A slow first-order branch exists only in a suitable overdamped sector; otherwise more variables are required.
Semiclassical linear response	Renormalized local terms and a retarded tensor kernel	A possible microscopic parent after constraint-compatible projection and verified scale separation.
Hamilton–Jacobi/WKB with GUP	Tunnelling on a specified black-hole geometry	Uses a different meaning of “semiclassical”; it does not itself compute reciprocal stress backreaction.
Present response closure	Two assumed relaxation poles and a chosen nonlinear completion	Gives conditional stability and bifurcation statements; no background expansion or data fit has been calculated.

In particular, Ali, Xia, and Babar [35] study Hawking temperature and thermodynamic corrections for a black hole in quintessence matter using a Hamilton–Jacobi/WKB construction with a generalized uncertainty principle. This is not an R^2 FLRW calculation or a solution of the renormalized stress-tensor backreaction equation. A quantitative com-

parison would require an additional computation of the stress response and its effect on the geometry. The common interest in quantum effects on curved backgrounds does not make the two approximations interchangeable.

The approach also differs from reheating/preheating calculations [36], where production and backreaction are tied to a specified evolving field sector. To confront observations, the present closure would need a reconstruction of $H(t)$ and the metric perturbations, physically calibrated R_* , T_* , state-dependent coefficients, and predictions for observables such as an expansion history or perturbation spectrum. The mathematical amplitudes alone supply none of those observables. The standard matrix threshold and normal-form methods are not presented as independent theoretical innovations.

9.3. Scope of the Cosmogenic Interpretation

A growing projected mode is neither particle creation by definition nor a proof of geodesic incompleteness. Conversely, bounded C and M do not prove boundedness of every curvature invariant or completeness of the reconstructed spacetime. In particular, $C = 0$ means no perturbation in the chosen projection, not a “pre-geometric” state. The framework presupposes an existing classical background, a quantum state, and a meaningful averaging prescription.

The homogeneous theorem describes the long-time alternatives through the equilibrium polynomial (33), the branches (30), and the convergence proof. Above threshold the stable set of the saddle remains an exceptional alternative to convergence to a nonzero branch. Physical use of any large-amplitude equilibrium must additionally check that it lies inside the amplitude and effective-theory range: boundedness in a polynomial model is not a guarantee of microscopic validity.

10. Conclusions

The reduced system supplies an exact conditional stability test once a conserved, constrained semiclassical response has been approximated by two suitable relaxation modes. Its cubic homogeneous completion is a gradient system with bounded trajectories converging to equilibria; the first spatial bifurcation on a fixed connected Neumann domain is homogeneous. Higher odd terms and a constant bias have distinct, explicitly derived effects on the local branches. These conclusions require neither an automatic runaway nor a singular cosmological interpretation.

The outstanding physical task is to compute the projected renormalized susceptibility and the geometric response for a specified field, state, and background, quantify the approximation error, and check whether any relevant root remains within the reliable frequency and amplitude range. The conditional FLRW construction identifies the required coefficients but does not complete that calculation. Until it is completed, the framework remains a response-level model with rigorous internal dynamics and no claimed observational prediction.

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